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Lie 2-Algebras

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by

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# ABSTRACT OF THE DISSERTATION

Lie 2-Algebras

by

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We categorify the theory of Lie algebras beginning with a new notion of categorified vector space, or ‘2-vector space’, which we define as an internal category in  $\mathbf{Vect}$ , the category of vector spaces. We then define a ‘semistrict Lie 2-algebra’ to be a 2-vector space  $L$  equipped with a skew-symmetric bilinear functor  $[\cdot, \cdot]: L \times L \rightarrow L$  satisfying the Jacobi identity up to a completely antisymmetric trilinear natural transformation called the ‘Jacobiator’, which in turn must satisfy a certain law of its own. We construct a 2-category of semistrict Lie 2-algebras and show that it is 2-equivalent to the 2-category of ‘2-term  $L_\infty$ -algebras’ in the sense of Stasheff. We also investigate strict and skeletal Lie 2-algebras. We show how to obtain the strict ones from Lie 2-groups and we use the skeletal ones to classify Lie 2-algebras in terms of 3rd cohomology classes in Lie algebra cohomology. This classification allows us to construct for any finite-dimensional Lie algebra  $\mathfrak{g}$  a canonical 1-parameter family of Lie 2-algebras  $\mathfrak{g}_\hbar$  which reduces to  $\mathfrak{g}$  at  $\hbar = 0$ .

We then explore the relationship between Lie algebras and algebraic structures called ‘quandles’. A quandle is a set  $Q$  equipped with two binary operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$  satisfying axioms that capture the essential properties of the operations of conjugation in a group and algebraically encode the three Reidemeister moves. Indeed, we describe the relation to groups and show that quandles give invariants of braids. We further show that both Lie algebras and quandles give solutions of the Yang–Baxter equation, and explain how conjugation plays a prominent role in the both the theories of Lie algebras and quandles. Inspired by these commonalities, we provide a novel, conceptual passage from Lie groups to Lie algebras using the language of quandles. Moreover, we propose relationships between higher Lie theory and higher-dimensional braid theory. We conclude with evidence of this connection by proving that any semistrict Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation, which is the higher-dimensional analog of the Yang–Baxter equation.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Semistrict Lie 2-algebras</b>	<b>7</b>
2.1	Internal Categories . . . . .	9
2.2	2-Vector spaces . . . . .	14
2.3	Semistrict Lie 2-algebras . . . . .	25
2.3.1	Definitions . . . . .	25
2.3.2	$L_\infty$ -algebras . . . . .	29
2.4	Strict Lie 2-algebras . . . . .	40
2.5	Skeletal Lie 2-algebras . . . . .	42
<b>3</b>	<b>Lie Theory, Quandles and Braids</b>	<b>48</b>
3.1	Shelves, Racks, Spindles and Quandles . . . . .	51
3.1.1	Definitions and Relation to Groups . . . . .	51
3.1.2	Relation to Topology . . . . .	54
3.1.3	Braid and Framed Braid Groups and Monoids . . . . .	62
3.1.4	Internalization . . . . .	72
3.2	From Lie Groups to Lie Algebras . . . . .	79
3.2.1	Cojets . . . . .	80
3.2.2	Special Coalgebras . . . . .	85
3.2.3	Unital Spindles . . . . .	89
3.2.4	From Unital Spindles to Lie Algebras . . . . .	92
3.3	Lie 2-algebras, 2-Quandles and 2-Braids . . . . .	98
	<b>Bibliography</b>	<b>110</b>

# Chapter 1

## Introduction

Higher-dimensional algebra is the study of generalizations of algebraic concepts obtained through a process called ‘categorification’. In the mid-1990’s, Crane [C, CF] coined this term to refer to the process of developing category-theoretic analogs of set-theoretic concepts. In this process we replace elements with objects, sets with categories, and functions with functors. We replace equations between elements by isomorphisms between objects, and replace equations between functions by natural isomorphisms between functors. Finally, we require that these isomorphisms satisfy equations of their own, called coherence laws. Finding the correct coherence laws is often the most difficult aspect of this generalization process. For example, the category  $\mathbf{FinSet}$  of finite sets together with functions is a categorification of the set of natural numbers. The functions of sum and product in  $\mathbb{N}$  are replaced by the functors disjoint union and Cartesian product of finite sets. The equational laws satisfied by addition and multiplication in  $\mathbb{N}$ , such as commutativity, associativity, and distributivity, now hold for disjoint union and Cartesian product, but only up to natural isomorphism. For instance, the associative law

$$(xy)z = x(yz)$$

is replaced by a natural isomorphism called the *associator*:

$$a_{x,y,z}: (xy)z \xrightarrow{\sim} x(yz),$$

which we then require to satisfy a coherence law known as the pentagon identity:

$$\begin{array}{ccccc}
 & & (wx)(yz) & & \\
 & \nearrow^{a_{(wx),y,z}} & & \searrow_{a_{w,x,(yz)}} & \\
 ((wx)y)z & & & & w(x(yz)) \\
 & \searrow_{a_{w,x,y} \times 1_z} & & \nearrow_{1_w \times a_{x,y,z}} & \\
 & (w(xy))z & \xrightarrow{a_{w,(xy),z}} & w((xy)z) & 
 \end{array}$$

Ultimately, by iterating this process, mathematicians wish to obtain and apply the  $n$ -categorical generalizations of as many mathematical concepts as possible to strengthen and simplify the connections between different subfields of mathematics.

Perhaps the greatest strength of categorification is that it allows us to refine our concept of ‘sameness’ by enabling us to distinguish between equality and isomorphism. In a set, two elements are either the same or different, while in a category, two objects can be ‘the same in a way’ while remaining different. That is, they can be isomorphic, but not equal. Better still, we are able to explicitly keep track of how two objects are the same: the isomorphism itself. Moreover, two objects can be the same in multiple ways, since there can be various different isomorphisms between them. An object can even be the same as itself in multiple ways: this leads to the concept of *symmetry*, since the automorphisms of an object form a group, its ‘symmetry group’. This more careful consideration of the notion of sameness is the reason that categorification plays an increasingly important role not only in mathematics, but also in physics and computer science, where symmetry plays a significant role and a precise treatment of the notion of sameness is crucial.

As an example, consider the problem of determining whether two sets are isomorphic. We know that we can solve this problem easily by simply counting the number of elements in each set. By comparing the resulting numbers, we will know that our sets are isomorphic, but we will not know *how* they are isomorphic. In examples, there will often be a specific isomorphism demonstrating that the two sets are isomorphic. Categorification is an attempt to preserve this type of information.

At various stages in the process of categorifying, we are forced to make certain decisions about the laws governing our objects and morphisms. On the one hand, we can insist that these laws hold ‘on the nose’, that is, that they hold *strictly* as equational laws. In this approach, the issue of coherence laws does not arise. On the other hand, we can impose these laws only *up to isomorphism*, with our isomorphisms then satisfying certain coherence laws. We refer to this second method as ‘weakening’. Categories constructed in this second way tend to be more appealing because they typically arise naturally in applications.

Typically when categorifying, we are attempting to find a category-theoretic analog of some algebraic structure defined in the category of sets, so we seek to create a hybrid of two notions: a category and that algebraic structure. Here a technique invented by Ehresmann [E], called ‘internalization’, can serve as a useful first step.

This process allows one to take concepts that were defined in the category  $\mathbf{Set}$  and transport them to other categories. For example, the concepts of ‘group’ and ‘(small) category’ live in the world of sets, but they can actually live in other categories as well! Internalizing a concept consists of first expressing it completely in terms of commutative diagrams and then interpreting those diagrams in some ambient category,  $K$ . In order to define an ‘internal group’ or ‘group in  $K$ ,’  $K$  must have finite products, while the definition of an ‘internal category’ or ‘category in  $K$ ’ requires that  $K$  have finite limits. The notion of internalization tends to be somewhat confusing the first time around, so it is a beneficial exercise to convince yourself that when  $K = \mathbf{Set}$ , these reduce to the usual definitions of a group and category.

A quick and easy way to categorify an algebraic structure is to internalize it taking  $K$  to be  $\mathbf{Cat}$ , the category whose objects are (small) categories and whose morphisms

are functors. Unfortunately, this method does not weaken the concepts being categorified: equations between functions are promoted to equations between functors, not natural isomorphisms, and thus coherence laws are not introduced. So, while we make extensive use of internalization in this work, we still must then weaken various concepts — notably the concept of a Lie algebra — in a somewhat *ad hoc* way.

One should not be tempted to think that categorification is simply generalization for its own sake! For example, group theory serves as a powerful tool in all branches of science where symmetry plays a role. But, in many situations where we are tempted to use groups, it is actually more natural to use a richer sort of structure, where in addition to group elements describing symmetries, we also have isomorphisms between these, describing *symmetries between symmetries*. This sort of structure is a categorified group, or ‘2-group’. The theory of 2-groups, or ‘higher-dimensional group theory’, dates back to the mid 1940’s and has a complicated history, which is briefly sketched in the introduction to Baez and Lauda’s review article on 2-groups [BLau]: *Higher Dimensional Algebra V: 2-groups*, henceforth denoted as HDA5.

The first sort of categorified group is a *strict* 2-group — one in which all the laws hold on the nose as equations. As one might expect, strict 2-groups can be defined exactly as one would hope: using internalization! They are precisely groups in  $\mathbf{Cat}$ , so they are often called ‘categorical groups’. They have numerous applications to homotopy theory [Bro, BS], topological quantum field theory [Y], nonabelian cohomology [Bre, Bre2, Gi], the theory of nonabelian gerbes [Bre2, BreM], categorified gauge field theory [At, B, GP, Pf], and even quantum gravity [CS, CY]. However, the strict notion is not the best for all applications, so a weaker concept of ‘coherent 2-group’ has also been introduced — originally by Sinh [S], who used the term ‘*gr*-category’. A careful introduction to coherent 2-groups can be found in HDA5 [BLau].

In recent years, thanks to the work of Ronald Brown [Bro2], 2-groups have secured their place in the mathematical, and more recently, physical, literature as useful and important objects. As an example, the concept of ‘Lie 2-group’ becomes especially important in applications of 2-groups to geometry and physics, in particular, to gauge theory. A Lie 2-group roughly amounts to a 2-group in which the set of objects and the set of morphisms are manifolds, and all relevant maps are smooth. Until recently, only strict Lie 2-groups had been defined [B]. However, in HDA5 [BLau], the concept of a coherent Lie 2-group was introduced, which we hope will contribute to the study of generalized gauge theories. Moreover, we suspect that just as Lie groups and Lie algebras arise wherever differential geometry and symmetry appear, the same will be true for Lie 2-groups and ‘Lie 2-algebras’. That is, Lie 2-algebras will ideally contribute to geometry and physics much in the same way as Lie 2-groups, and will hopefully have applications in situations where the Jacobi identity need not hold on the nose.

The goal of this work is to develop and explore the theory of categorified Lie algebras, or ‘Lie 2-algebras’. Just as every Lie algebra has an underlying vector space, a Lie 2-algebra will be a ‘2-vector space’ equipped with extra structure. A 2-vector space blends together the notions of category and vector space, so, roughly speaking, it is a category where everything is linear. More precisely, we use internalization to define a 2-vector space to be a category in  $\mathbf{Vect}$ , the category of vector spaces. Then, to obtain the notion of a

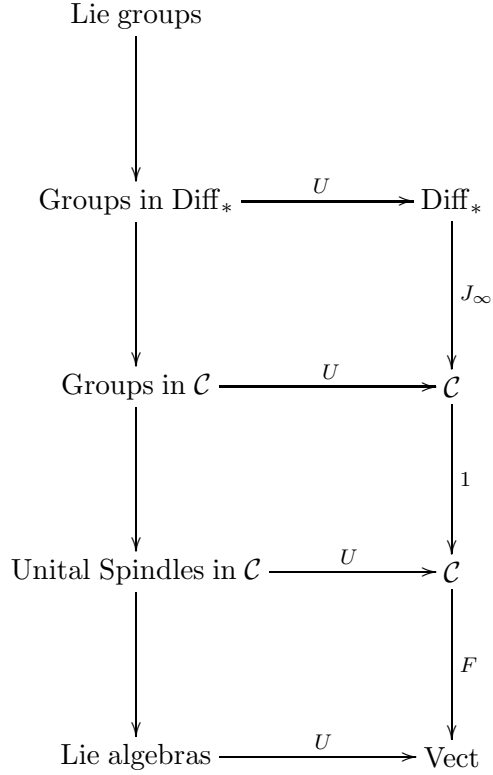


Lie 2-algebra, we start with a 2-vector space and equip it with a weakened version of the structure of a Lie algebra. That is, instead of a bracket function, a Lie 2-algebra has a bracket *functor*, which we require to be ‘linear’. We then weaken the Jacobi identity so that now it holds only *up to a ‘linear’ natural isomorphism*, which we call the ‘Jacobiator’. Following the Tao of weakening, we require that the Jacobiator satisfy an identity of its own, which we call the ‘Jacobiator identity’.

The theory of semistrict Lie 2-algebras is investigated in Chapter 2 in a presentation parallel to the development of the theory of 2-groups in HDA5 [BLau]. In particular, we show that just as coherent 2-groups were classified up to equivalence using group cohomology, semistrict Lie 2-algebras can be classified up to equivalence using Lie algebra cohomology. This classification allows us to construct a 1-parameter family of Lie 2-algebras  $\mathfrak{g}_\hbar$  deforming any finite-dimensional Lie algebra, which are especially interesting when  $\mathfrak{g}$  is semisimple, and may be related to affine Lie algebras and quantum groups. Furthermore, we demonstrate the relationship between semistrict Lie 2-algebras and special  $L_\infty$ -algebras, or sh-Lie algebras, which are generalizations of Lie algebras defined by Stasheff [SS] that blend together the notions of Lie algebra and chain complex. Much of the content of this chapter has already appeared in a separate paper coauthored with John Baez [BC], *Higher Dimensional Algebra VI: Lie 2-algebras*, henceforth denoted as HDA6.

In Chapter 3 we describe a modern approach to obtaining the Lie algebra of a Lie group, a result that holds a prominent place in differential geometry, using algebraic structures known as ‘quandles’. We begin with definitions and examples of quandles and continue by illustrating their relationship to group theory and topology, in particular, to the three Reidemeister moves. In fact, we show that quandles and Lie algebras give solutions to the Yang–Baxter equation. We also demonstrate how quandles give invariants of braids.

Our road map for the description of this new, conceptual, passage of Lie group to Lie algebra takes the following form:



We explain this diagram thoroughly by recalling the notion of a ‘jet’, introducing the notion of a ‘special coalgebra’, and describing their roles in this process. This chapter is a warmup for a future attempt to obtain the Lie 2-algebra of a Lie 2-group. In the final section of this chapter, we outline what we have done, and what remains to be done, in order to achieve this ‘higher’ result.

These two chapters are united by the fact that both Lie algebras and quandles give solutions of the Yang–Baxter equation. Furthermore, the theory of conjugation plays a prominent role in both the theories of Lie algebras and quandles, which we describe in detail in Chapter 3. These observations inspired our novel description of the passage from a Lie group to its Lie algebra. In both chapters, we emphasize the relationship between algebra and topology, which motivates the relationship between higher-dimensional algebra and higher-dimensional topology. While we have only briefly sketched the contents of these chapters here, more details can be found at the beginning of Chapter 2 and Chapter 3.

We conclude in Section 3.3 by outlining our future goals related to this work, most importantly being the task of showing that every Lie 2-group has a Lie 2-algebra. Furthermore, we offer our guesses for how the relationship between Lie theory and braids should generate a relationship between higher Lie theory and higher-dimensional braid theory. In particular, as evidence of this relationship, we prove that just as a Lie algebra gives a solution of the Yang–Baxter equation, or third Reidemeister move, a Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation, which is the higher-dimensional analog of the Yang–Baxter equation. This result, while beautiful in its own right, also suggests that the process of passing from Lie groups to Lie algebras given in Section 3.2

may categorify, providing us with the outcome we desire.

## Chapter 2

# Semistrict Lie 2-algebras

Just as the concept of Lie group can be categorified to obtain various concepts of ‘Lie 2-group’, one can categorify the concept of Lie algebra to obtain various concepts of ‘Lie 2-algebra’. As mentioned in the Introduction, a Lie 2-group, roughly speaking, is a categorified group where everything in sight is smooth. Since we are going to categorify the notion of a Lie algebra in approximately the same way, we begin by recalling the manner in which we categorify a group.

We begin by using the technique of internalization, which we recalled in the Introduction. That is, given a group  $G$ , we replace the underlying *set* with a *category*, and the multiplication *function*  $m: G \times G \rightarrow G$  with a multiplication *functor*  $m: G \otimes G \rightarrow G$ . And this is where we stop if we want to describe the notion of a *strict* 2-group. Recall from the Introduction that when we define a strict concept, all laws hold strictly as equations. Thus, the issue of coherence laws does not arise. However, if we choose to *weaken* our 2-group, we impose the group laws only *up to isomorphism*, and then require these isomorphisms to satisfy certain laws of their own. Thus, to weaken the concept of 2-group, we replace the *equation* expressing the associative law by an *isomorphism* called the ‘associator’:

$$a_{x,y,z}: (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z),$$

which we then require to satisfy a coherence law known as the ‘pentagon identity’:

$$\begin{array}{ccccc}
 & & (w \otimes x) \otimes (y \otimes z) & & \\
 & \nearrow^{a_{(w \otimes x), y, z}} & & \nwarrow_{a_{w, x, (y \otimes z)}} & \\
 ((w \otimes x) \otimes y) \otimes z & & & & w \otimes (x \otimes (y \otimes z)) \\
 & \searrow_{a_{w, x, y} \otimes 1_z} & & \nearrow_{1_w \otimes a_{x, y, z}} & \\
 (w \otimes (x \otimes y)) \otimes z & \xrightarrow{a_{w, (x \otimes y), z}} & w \otimes ((x \otimes y) \otimes z) & & 
 \end{array}$$

Similarly, we replace the equations expressing the left and right unit laws

$$1 \otimes x = x, \quad x \otimes 1 = x$$

by isomorphisms

$$\ell_x: 1 \otimes x \xrightarrow{\sim} x, \quad r_x: x \otimes 1 \xrightarrow{\sim} x$$

To obtain the notion of a ‘coherent’ 2-group, we weaken further by replacing the equations

$$x \otimes x^{-1} = 1, \quad x^{-1} \otimes x = 1$$

by isomorphisms called the ‘unit’ and ‘counit’. Thus, instead of an inverse in the strict sense, the object  $x$  only has a specified ‘weak inverse’.

We now consider categorifying the notion of a Lie algebra. In the theory of Lie algebras, the analog to the associative law is the Jacobi identity. Therefore, in analogy to our discussion above, to obtain a ‘Lie 2-algebra, we will replace the Jacobi identity by an isomorphism that we call the *Jacobiator*, which, we will see, satisfies an interesting new law of its own. Just as the pentagon equation can be traced back to work done by Stasheff, we will see that the coherence law for the Jacobiator is related to his work on  $L_\infty$ -algebras, also known as strongly homotopy Lie algebras [LS, SS]. This demonstrates yet again the close connection between categorification and homotopy theory.

Since our goal is to categorify a Lie algebra, which will be a mixture of a Lie algebra and category, we first consider the notion of categorified vector space. Just as a Lie algebra has an underlying vector space, a Lie 2-algebra will have an underlying ‘2-vector space’. A 2-vector space will blend together the concept of a category with that of a vector space. Thus, we begin in Section 2.1 by reviewing the theory of internal categories, which we will use to create such a concept. In Section 2.2 we focus specifically on categories in  $\mathbf{Vect}$ , the category of vector spaces. We boldly call these ‘2-vector spaces’, despite the fact that this term is already used to refer to a very different categorification of the concept of vector space [KV], for it is our contention that our 2-vector spaces lead to a more interesting version of categorified linear algebra than the traditional ones. For example, the tangent space at the identity of a Lie 2-group is a 2-vector space of our sort, and this gives a canonical representation of the Lie 2-group: its ‘adjoint representation’. This is contrast to the phenomenon observed by Barrett and Mackaay [BM], namely that Lie 2-groups have few interesting representations on the traditional sort of 2-vector space. One reason for the difference is that the traditional 2-vector spaces do not have a way to ‘subtract’ objects, while ours do. This will be especially important for finding examples of Lie 2-algebras, since we often wish to set  $[x, y] = xy - yx$ .

At this point we should admit that our 2-vector spaces are far from novel entities! In fact, a category in  $\mathbf{Vect}$  is secretly just the same as a 2-term chain complex of vector spaces. While the idea behind this correspondence goes back to Grothendieck [G], and is by now well-known to category-theorists, we describe it carefully in Proposition 8, because it is crucial for relating ‘categorified linear algebra’ to more familiar ideas from homological algebra.

In Section 2.3.1 we introduce the key concept of ‘semistrict Lie 2-algebra’. Roughly speaking, this is a 2-vector space  $L$  equipped with a bilinear functor

$$[\cdot, \cdot]: L \times L \rightarrow L,$$

the Lie bracket, that is skew-symmetric and satisfies the Jacobi identity up to a completely antisymmetric trilinear natural isomorphism, the ‘Jacobiator’ — which in turn is required

to satisfy a law of its own, the ‘Jacobiator identity’. Since we do not weaken the equation  $[x, y] = -[y, x]$  to an isomorphism, we do not reach the more general concept of ‘weak Lie 2-algebra’: this remains a task for the future, which we describe in further detail in Section 3.3 of the next chapter.

In Section 2.3.2, we recall the definition of an  $L_\infty$ -algebra. Briefly, this is a chain complex  $V$  of vector spaces equipped with a bilinear skew-symmetric operation  $[\cdot, \cdot]: V \times V \rightarrow V$  which satisfies the Jacobi identity up to an infinite tower of chain homotopies. We construct a 2-category of ‘2-term’  $L_\infty$ -algebras, that is, those with  $V_i = \{0\}$  except for  $i = 0, 1$ . Finally, we show this 2-category is equivalent to the previously defined 2-category of semistrict Lie 2-algebras.

In the next two sections we study *strict* and *skeletal* Lie 2-algebras, the former being those where the Jacobi identity holds ‘on the nose’, while in the latter, isomorphisms exist only between identical objects. Section 2.4 consists of an introduction to strict Lie 2-algebras and strict Lie 2-groups, together with the process for obtaining the strict Lie 2-algebra of a strict Lie 2-group. Section 2.5 begins with an exposition of Lie algebra cohomology and its relationship to skeletal Lie 2-algebras. We then show that Lie 2-algebras can be classified (up to equivalence) in terms of a Lie algebra  $\mathfrak{g}$ , a representation of  $\mathfrak{g}$  on a vector space  $V$ , and an element of the Lie algebra cohomology group  $H^3(\mathfrak{g}, V)$ . With the help of this result, we construct from any finite-dimensional Lie algebra  $\mathfrak{g}$  a canonical 1-parameter family of Lie 2-algebras  $\mathfrak{g}_\hbar$  which reduces to  $\mathfrak{g}$  at  $\hbar = 0$ . This is a new way of deforming a Lie algebra, in which the Jacobi identity is weakened in a manner that depends on the parameter  $\hbar$ . It is natural to speculate that this deformation is somehow related to the theory of quantum groups and affine Lie algebras. However, we have only a little evidence for this speculation at present.

**Note:** In all that follows, we denote the composite of morphisms  $f: x \rightarrow y$  and  $g: y \rightarrow z$  as  $fg: x \rightarrow z$ . All 2-categories and 2-functors referred to are *strict*, though sometimes we include the word ‘strict’ to emphasize this fact. We denote vertical composition of 2-morphisms by juxtaposition; we denote horizontal composition and whiskering by the symbol  $\circ$ .

## 2.1 Internal Categories

In order to create a hybrid of the notions of a vector space and a category in the next section, we will use the technique of internalization to blend together these concepts. That is, we need the concept of an ‘internal category’, also called a ‘category object’, within some category. The idea is that given a category  $K$ , we obtain the definition of a ‘category internal to  $K$ ’, which we call ‘category in  $K$ ’ for short, by expressing the definition of a usual (small) category completely in terms of commutative diagrams and then interpreting those diagrams within  $K$ . The same idea allows us to define functors and natural transformations in  $K$ , at least if  $K$  has properties sufficiently resembling those of the category of sets.

Internal categories were introduced by Ehresmann [E] in the 1960s, and by now they are a standard part of category theory [Bo]. However, since not all readers may be familiar with them, for the sake of a self-contained treatment we start with the basic definitions.

**Definition 1.** Let  $K$  be a category. A **category internal to  $K$**  or **category in  $K$** , say  $X$ , consists of:

- an **object of objects**  $X_0 \in K$ ,
- an **object of morphisms**  $X_1 \in K$ ,

together with

- **source and target morphisms**  $s, t: X_1 \rightarrow X_0$ ,
- a **identity-assigning morphism**  $i: X_0 \rightarrow X_1$ ,
- a **composition morphism**  $\circ: X_1 \times_{X_0} X_1 \rightarrow X_1$

such that the following diagrams commute, expressing the usual category laws:

- laws specifying the source and target of identity morphisms:

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ & \searrow 1 & \downarrow s \\ & & X_0 \end{array} \quad \begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ & \searrow 1 & \downarrow t \\ & & X_0 \end{array}$$

- laws specifying the source and target of composite morphisms:

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1 \\ \downarrow p_1 & & \downarrow s \\ X_1 & \xrightarrow{s} & X_0 \end{array} \quad \begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1 \\ \downarrow p_2 & & \downarrow t \\ X_1 & \xrightarrow{t} & X_0 \end{array}$$

- the associative law for composition of morphisms:

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 \times_{X_0} X_1 & \xrightarrow{\circ \times_{X_0} 1} & X_1 \times_{X_0} X_1 \\ \downarrow 1 \times_{X_0} \circ & & \downarrow \circ \\ X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1 \end{array}$$

- the left and right unit laws for composition of morphisms:

$$\begin{array}{ccccc} X_0 \times_{X_0} X_1 & \xrightarrow{i \times 1} & X_1 \times_{X_0} X_1 & \xleftarrow{1 \times i} & X_1 \times_{X_0} X_0 \\ & \searrow p_2 & \downarrow \circ & \swarrow p_1 & \\ & & X_1 & & \end{array}$$

The pullbacks referred to in the above definition should be clear from the usual definition of category; for instance, composition is defined on pairs of morphisms where the target of the first is the source of the second, so the pullback  $X_1 \times_{X_0} X_1$  is defined via the square

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{p_1} & X_1 \\ \downarrow p_2 & & \downarrow s \\ X_1 & \xrightarrow{t} & X_0 \end{array}$$

Notice that inherent to this definition is the assumption that the pullbacks involved actually exist. This holds automatically when the ‘ambient category’  $K$  has finite limits, but there are some important examples such as  $K = \text{Diff}$  where this is not the case. Throughout this work, all of the categories considered have finite limits:

- *Set*, the category whose objects are sets and whose morphisms are functions.
- *Vect*, the category whose objects are vector spaces over the field  $k$  and whose morphisms are linear functions.
- *Grp*, the category whose objects are groups and whose morphisms are homomorphisms.
- *Cat*, the category whose objects are small categories and whose morphisms are functors.
- *LieGrp*, the category whose objects are Lie groups and whose morphisms are Lie group homomorphisms.
- *LieAlg*, the category whose objects are Lie algebras over the field  $k$  and whose morphisms are Lie algebra homomorphisms.

Having defined ‘categories in  $K$ ’, we can now internalize the notions of functor and natural transformation in a similar manner to obtain functors and natural transformations in  $K$ . We shall use these to construct a 2-category  $K\text{Cat}$  consisting of categories, functors, and natural transformations in  $K$ .

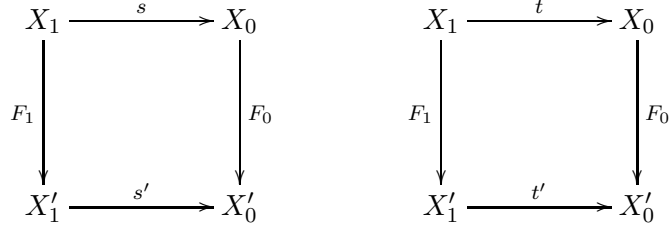
**Definition 2.** *Let  $K$  be a category. Given categories  $X$  and  $X'$  in  $K$ , an **internal functor** or **functor in  $K$**  between them, say  $F: X \rightarrow X'$ , consists of:*

- *a morphism  $F_0: X_0 \rightarrow X'_0$ ,*
- *a morphism  $F_1: X_1 \rightarrow X'_1$*

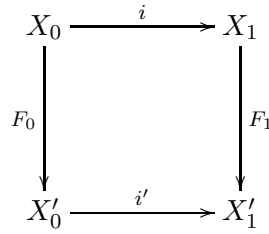
*such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:*



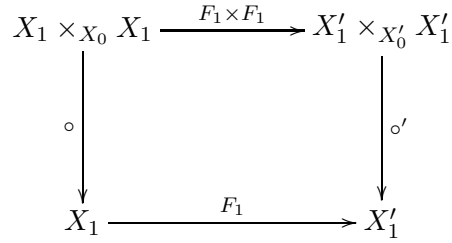
- *preservation of source and target:*



- *preservation of identity morphisms:*



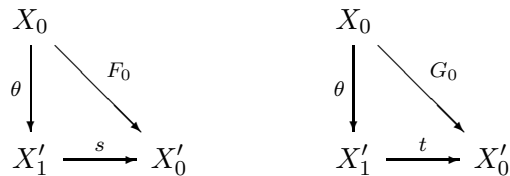
- *preservation of composite morphisms:*



Given two functors  $F: X \rightarrow X'$  and  $G: X' \rightarrow X''$  in some category  $K$ , we define their composite  $FG: X \rightarrow X''$  by taking  $(FG)_0 = F_0G_0$  and  $(FG)_1 = F_1G_1$ . Similarly, we define the identity functor in  $K$ ,  $1_X: X \rightarrow X$ , by taking  $(1_X)_0 = 1_{X_0}$  and  $(1_X)_1 = 1_{X_1}$ . Showing that the composite  $FG$  satisfies the diagrams of Definition 2 is a straightforward computation. We now consider morphisms between these functors in  $K$ :

**Definition 3.** *Let  $K$  be a category. Given two functors  $F, G: X \rightarrow X'$  in  $K$ , an **internal natural transformation** or **natural transformation in  $K$**  between them, say  $\theta: F \Rightarrow G$ , is a morphism  $\theta: X_0 \rightarrow X'_1$  for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:*

- *laws specifying the source and target of a natural transformation:*



- the commutative square law:

$$\begin{array}{ccc}
X_1 & \xrightarrow{\Delta(s\theta \times G)} & X'_1 \times_{X'_0} X'_1 \\
\Delta(F \times t\theta) \downarrow & & \downarrow \circ' \\
X'_1 \times_{X'_0} X'_1 & \xrightarrow{\circ'} & X'_1
\end{array}$$

Just like ordinary natural transformations, natural transformations in  $K$  may be composed in two different, but commuting, ways. First, let  $X$  and  $X'$  be categories in  $K$  and let  $F, G, H: X \rightarrow X'$  be functors in  $K$ . If  $\theta: F \Rightarrow G$  and  $\tau: G \Rightarrow H$  are natural transformations in  $K$ , we define their **vertical** composite,  $\theta\tau: F \Rightarrow H$ , by

$$\theta\tau := \Delta(\theta \times \tau) \circ'.$$

The reader can check that when  $K = \text{Cat}$  this reduces to the usual definition of vertical composition. We can represent this composite pictorially as:

$$\begin{array}{c}
\begin{array}{ccc}
& F & \\
X & \xrightarrow{\quad} & X' \\
& \Downarrow \theta\tau & \\
& H & 
\end{array}
=
\begin{array}{ccc}
& F & \\
X & \xrightarrow{G} & X' \\
& \Downarrow \tau & \\
& H & 
\end{array}
\end{array}$$

Next, let  $X, X', X''$  be categories in  $K$  and let  $F, G: X \rightarrow X'$  and  $F', G': X' \rightarrow X''$  be functors in  $K$ . If  $\theta: F \Rightarrow G$  and  $\theta': F' \Rightarrow G'$  are natural transformations in  $K$ , we define their **horizontal composite**,  $\theta \circ \theta': FF' \Rightarrow GG'$ , in either of two equivalent ways:

$$\begin{aligned}
\theta \circ \theta' &:= \Delta(F_0 \times \theta)(\theta' \times G'_1) \circ' \\
&= \Delta(\theta \times G_0)(F'_1 \times \theta') \circ'.
\end{aligned}$$

Again, this reduces to the usual definition when  $K = \text{Cat}$ . The horizontal composite can be depicted as:

$$\begin{array}{ccc}
\begin{array}{ccc}
& FF' & \\
X & \xrightarrow{\quad} & X'' \\
& \Downarrow \theta \circ \theta' & \\
& GG' & 
\end{array}
=
\begin{array}{ccccc}
& F & & F' & \\
X & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & X'' \\
& \Downarrow \theta & & \Downarrow \theta' & \\
& G & & G' & 
\end{array}
\end{array}$$

It is routine to check that these composites are again natural transformations in  $K$ . Finally, given a functor  $F: X \rightarrow X'$  in  $K$ , the identity natural transformation  $1_F: F \Rightarrow F$  in  $K$  is given by  $1_F = F_0 i$ .

We now have all the ingredients of a 2-category:

**Proposition 4.** *Let  $K$  be a category. Then there exists a strict 2-category  $\mathbf{KCat}$  with categories in  $K$  as objects, functors in  $K$  as morphisms, and natural transformations in  $K$  as 2-morphisms, with composition and identities defined as above.*

**Proof.** It is straightforward to check that all the axioms of a 2-category hold; this result goes back to Ehresmann [E].  $\square$

We now consider internal categories in  $\mathbf{Vect}$ .

## 2.2 2-Vector spaces

Since our goal is to categorify the concept of a Lie algebra, we must first categorify the concept of a vector space. A categorified vector space, or ‘2-vector space’, should combine the notions of category and vector space, and should therefore be a category where everything in sight is linear. That is, it will be a category with structure analogous to that of a vector space, with functors replacing the usual vector space operations. Kapranov and Voevodsky [KV] implemented this idea by taking a finite-dimensional 2-vector space to be a category of the form  $\mathbf{Vect}^n$ , in analogy to how every finite-dimensional vector space is of the form  $k^n$ . While this idea is useful in contexts such as topological field theory [L] and group representation theory [B2], it has its limitations, which arise from the fact that there is no notion of subtraction in these 2-vector spaces.

Here we instead define a 2-vector space to be a category in  $\mathbf{Vect}$ . Just as the main ingredient of a Lie algebra is a vector space, a Lie 2-algebra will have an underlying 2-vector space of this sort. Thus, in this section we first define a 2-category of these 2-vector spaces. We then establish the relationship between these 2-vector spaces and 2-term chain complexes of vector spaces: that is, chain complexes having only two nonzero vector spaces. We conclude this section by developing some ‘categorified linear algebra’ — the bare minimum necessary for defining and working with Lie 2-algebras in the next section.

In the following we consider vector spaces over an arbitrary field,  $k$ .

**Definition 5.** *A 2-vector space is a category in  $\mathbf{Vect}$ .*

Thus, a 2-vector space  $V$  is a category with a vector space of objects  $V_0$  and a vector space of morphisms  $V_1$ , such that the source and target maps  $s, t: V_1 \rightarrow V_0$ , the identity-assigning map  $i: V_0 \rightarrow V_1$ , and the composition map  $\circ: V_1 \times_{V_0} V_1 \rightarrow V_1$  are all *linear*. As usual, we write a morphism as  $f: x \rightarrow y$  when  $s(f) = x$  and  $t(f) = y$ , and sometimes we write  $i(x)$  as  $1_x$ .

In fact, the structure of a 2-vector space is completely determined by the vector spaces  $V_0$  and  $V_1$  together with the source, target and identity-assigning maps. As the following lemma demonstrates, composition can always be expressed in terms of these, together with vector space addition:

**Lemma 6.** *When  $K = \mathbf{Vect}$ , one can omit all mention of composition in the definition of category in  $K$ , without any effect on the concept being defined.*

**Proof.** First, given vector spaces  $V_0, V_1$  and maps  $s, t: V_1 \rightarrow V_0$  and  $i: V_0 \rightarrow V_1$ , we will define a composition operation that satisfies the laws in Definition 1, obtaining a 2-vector space.

Given  $f \in V_1$ , we define the **arrow part** of  $f$ , denoted as  $\vec{f}$ , by

$$\vec{f} = f - i(s(f)).$$

Notice that  $\vec{f}$  is in the kernel of the source map since

$$s(f - i(s(f))) = s(f) - s(f) = 0.$$

While the source of  $\vec{f}$  is always zero, its target may be computed as follows:

$$t(\vec{f}) = t(f - i(s(f))) = t(f) - s(f).$$

The meaning of the arrow part becomes clearer if we write  $f: x \rightarrow y$  when  $s(f) = x$  and  $t(f) = y$ . Then, given any morphism  $f: x \rightarrow y$ , we have  $\vec{f}: 0 \rightarrow y - x$ . In short, taking the arrow part of  $f$  has the effect of ‘translating  $f$  to the origin’.

We can always recover any morphism from its arrow part together with its source, since  $f = \vec{f} + i(s(f))$ . We shall take advantage of this by identifying  $f: x \rightarrow y$  with the ordered pair  $(x, \vec{f})$ . Note that with this notation we have

$$s(x, \vec{f}) = x, \quad t(x, \vec{f}) = x + t(\vec{f}).$$

Using this notation, given morphisms  $f: x \rightarrow y$  and  $g: y \rightarrow z$ , we define their composite by

$$fg := (x, \vec{f} + \vec{g}),$$

or equivalently,

$$(x, \vec{f})(y, \vec{g}) := (x, \vec{f} + \vec{g}),$$

where the addition is the vector addition in  $V_1$ . It remains to show that with this composition, the diagrams of Definition 1 commute. The triangles specifying the source and target of the identity-assigning morphism do not involve composition. The second pair of diagrams commute since

$$s(fg) = x$$

and

$$t(fg) = x + t(\vec{f}) + t(\vec{g}) = x + (y - x) + (z - y) = z.$$

The associative law holds for composition because vector space addition is associative. Finally, the left unit law is satisfied since given  $f: x \rightarrow y$ ,

$$i(x)f = (x, 0)(x, \vec{f}) = (x, \vec{f}) = f$$

and similarly for the right unit law. We thus have a 2-vector space.

Conversely, given a category  $V$  in  $\mathbf{Vect}$ , we shall show that its composition must be defined by the formula given above. Suppose that  $(f, g) = ((x, \vec{f}), (y, \vec{g}))$  and  $(f', g') = ((x', \vec{f}'), (y', \vec{g}'))$  are composable pairs of morphisms in  $V_1$ . Since the source and target maps

are linear,  $(f + f', g + g')$  also forms a composable pair, and the linearity of composition gives

$$(f + f')(g + g') = fg + f'g'.$$

If we set  $g = 1_y$  and  $f' = 1_{y'}$ , the above equation becomes

$$(f + 1_{y'})(1_y + g') = f1_y + 1_{y'}g' = f + g'.$$

Expanding out the left hand side we obtain

$$((x, \vec{f}) + (y', 0))((y, 0) + (y', \vec{g}')) = (x + y', \vec{f})(y + y', \vec{g}'),$$

while the right hand side becomes

$$(x, \vec{f}) + (y, \vec{g}') = (x + y', \vec{f} + \vec{g}').$$

Thus we have  $(x + y', \vec{f})(y + y', \vec{g}') = (x + y', \vec{f} + \vec{g}')$ , so the formula for composition in an arbitrary 2-vector space must be given by

$$fg = (x, \vec{f})(y, \vec{g}) = (x, \vec{f} + \vec{g})$$

whenever  $(f, g)$  is a composable pair. This shows that we can leave out all reference to composition in the definition of ‘category in  $K$ ’ without any effect when  $K = \mathbf{Vect}$ .  $\square$

In order to simplify future arguments, we will often use only the elements of the above lemma to describe a 2-vector space.

We continue by defining the morphisms between 2-vector spaces:

**Definition 7.** *Given 2-vector spaces  $V$  and  $W$ , a **linear functor**  $F: V \rightarrow W$  is a functor in  $\mathbf{Vect}$  from  $V$  to  $W$ .*

For now we let  $2\mathbf{Vect}$  stand for the category of 2-vector spaces and linear functors between them; later we will make  $2\mathbf{Vect}$  into a 2-category.

The reader may already have noticed that a 2-vector space resembles a **2-term chain complex** of vector spaces: that is, a pair of vector spaces with a linear map between them, called the ‘differential’:

$$C_1 \xrightarrow{d} C_0.$$

In fact, this analogy is very precise. Moreover, it continues at the level of morphisms. A **chain map** between 2-term chain complexes, say  $\phi: C \rightarrow C'$ , is simply a pair of linear maps  $\phi_0: C_0 \rightarrow C'_0$  and  $\phi_1: C_1 \rightarrow C'_1$  that ‘preserves the differential’, meaning that the following square commutes:

$$\begin{array}{ccc} C_1 & \xrightarrow{d} & C_0 \\ \phi_1 \downarrow & & \downarrow \phi_0 \\ C'_1 & \xrightarrow{d'} & C'_0 \end{array}$$

There is a category  $2\mathbf{Term}$  whose objects are 2-term chain complexes and whose morphisms are chain maps. Moreover:

**Proposition 8.** *The categories  $2\text{Vect}$  and  $2\text{Term}$  are equivalent.*

**Proof.** We begin by introducing functors

$$S: 2\text{Vect} \rightarrow 2\text{Term}$$

and

$$T: 2\text{Term} \rightarrow 2\text{Vect}.$$

We first define  $S$ . Given a 2-vector space  $V$ , we define  $S(V) = C$  where  $C$  is the 2-term chain complex with

$$\begin{aligned} C_0 &= V_0, \\ C_1 &= \ker(s) \subseteq V_1, \\ d &= t|_{C_1}, \end{aligned}$$

and  $s, t: V_1 \rightarrow V_0$  are the source and target maps associated with the 2-vector space  $V$ . It remains to define  $S$  on morphisms. Let  $F: V \rightarrow V'$  be a linear functor and let  $S(V) = C$ ,  $S(V') = C'$ . We define  $S(F) = \phi$  where  $\phi$  is the chain map with  $\phi_0 = F_0$  and  $\phi_1 = F_1|_{C_1}$ . Note that  $\phi$  preserves the differential because  $F$  preserves the target map.

To show that  $S$  is a functor, we consider composable internal functors  $F$  and  $G$ :

$$V \xrightarrow{F} V' \xrightarrow{G} V''$$

where  $S(F) = \phi$  and  $S(G) = \tau$ . Then,  $S(FG) = \theta$  where  $\theta$  is the chain map with

$$\theta_0 = (FG)_0 = F_0 G_0 = \phi_0 \tau_0$$

and

$$\theta_1 = (FG)_1|_{C_1} = F_1 G_1|_{C_1} = \phi_1 \tau_1|_{C_1}$$

where the second equality in each of the above follows from the definition of the composite of linear functors. We thus have  $\theta = \phi\tau$ , so  $S$  preserves composition.

To show  $S$  preserves identities, consider a 2-vector space  $V$  with  $1_V: V \rightarrow V$ . If we let  $S(V) = C$ , then  $S(1_V) = \phi$  where  $\phi$  is the chain map with

$$\phi_0 = (1_V)_0 = 1_{V_0} = 1_{C_0} = 1_{S(V)_0}$$

and

$$\phi_1 = (1_V)_1|_{C_1} = 1_{V_1}|_{C_1} = 1_{C_1} = 1_{S(V)_1}$$

where we have used the definition of the identity functor in  $\text{Vect}$ . We thus have  $\phi = 1_C$ , so  $S$  preserves identities.

We now turn to the second functor,  $T$ . Given a 2-term chain complex  $C$ , we define  $T(C) = V$  where  $V$  is a 2-vector space with

$$\begin{aligned} V_0 &= C_0, \\ V_1 &= C_0 \oplus C_1. \end{aligned}$$

To completely specify  $V$  it suffices by Lemma 6 to specify linear maps  $s, t: V_1 \rightarrow V_0$  and  $i: V_0 \rightarrow V_1$  and check that  $s(i(x)) = t(i(x)) = x$  for all  $x \in V_0$ . To define  $s$  and  $t$ , we write any element  $f \in V_1$  as a pair  $(x, \vec{f}) \in C_0 \oplus C_1$  and set

$$\begin{aligned} s(f) &= s(x, \vec{f}) = x, \\ t(f) &= t(x, \vec{f}) = x + d\vec{f}. \end{aligned}$$

For  $i$ , we use the same notation and set

$$i(x) = (x, 0)$$

for all  $x \in V_0$ . Clearly  $s(i(x)) = t(i(x)) = x$ . Note also that with these definitions, the decomposition  $V_1 = C_0 \oplus C_1$  is precisely the decomposition of morphisms into their source and ‘arrow part’, as in the proof of Lemma 6. Moreover, given any morphism  $f = (x, \vec{f}) \in V_1$ , we have

$$t(f) - s(f) = d\vec{f}.$$

Next we define  $T$  on morphisms. Suppose  $\phi: C \rightarrow C'$  is a chain map between 2-term chain complexes:

$$\begin{array}{ccc} C_1 & \xrightarrow{d} & C_0 \\ \phi_1 \downarrow & & \downarrow \phi_0 \\ C'_1 & \xrightarrow{d'} & C'_0 \end{array}$$

Let  $T(C) = V$  and  $T(C') = V'$ . Then we define  $T(\phi) = F$  where  $F: V \rightarrow V'$  is the linear functor with  $F_0 = \phi_0$  and  $F_1 = \phi_0 \oplus \phi_1$ . To check that  $F$  really is a linear functor, note that it is linear on objects and morphisms. Moreover, it preserves the source and target, identity-assigning and composition maps because all these are defined in terms of addition and the differential in the chain complexes  $C$  and  $C'$ , and  $\phi$  is linear and preserves the differential.

To show that  $T$  is a functor, we consider composable chain maps  $\phi$  and  $\phi'$ :

$$C \xrightarrow{\phi} C' \xrightarrow{\phi'} C''$$

where  $T(\phi) = F$  and  $T(\phi') = G$ . Then, on one hand we have  $T(\phi\phi') = H$  where  $H$  is the linear functor with

$$H_0 = (\phi\phi')_0 = \phi_0\phi'_0$$

and

$$H_1 = (\phi\phi')_1 = \phi_0\phi'_1 \oplus \phi_1\phi'_1$$

On the other hand,  $T(\phi)T(\phi') = FG$  where  $FG$  is the linear functor with

$$(FG)_0 = F_0G_0 = \phi_0\phi'_0$$

and

$$(FG)_1 = F_1 G_1 = (\phi_0 \oplus \phi_1)(\phi'_0 \oplus \phi'_1) = \phi_0 \phi'_0 \oplus \phi_1 \phi'_1$$

which is equal to the linear functor  $H$ , so that  $T$  preserves composition.

To show  $T$  preserves identities, let  $C$  be a 2-term chain complex  $C$  with  $1_C: C \rightarrow C$ . Let  $T(C) = V$ . Then  $T(1_C) = F$  where  $F$  is the linear functor with

$$F_0 = (1_C)_0 = 1_{C_0} = 1_{V_0} = (1_V)_0 = 1_{T(C)_0}$$

and

$$F_1 = 1_{C_0} \oplus 1_{C_1} = 1_{C_0 \oplus C_1} = 1_{V_1} = (1_V)_1 = 1_{T(C)_1}$$

so that  $T$  preserves identities.

To show that  $S$  and  $T$  form an equivalence, we construct natural isomorphisms  $\alpha: ST \Rightarrow 1_{2\text{Vect}}$  and  $\beta: TS \Rightarrow 1_{2\text{Term}}$ .

To construct  $\alpha$ , consider a 2-vector space  $V$ . Applying  $S$  to  $V$  we obtain the 2-term chain complex

$$\ker(s) \xrightarrow{t|_{\ker(s)}} V_0.$$

Applying  $T$  to this result, we obtain a 2-vector space  $V'$  with the space  $V_0$  of objects and the space  $V_0 \oplus \ker(s)$  of morphisms. The source map for this 2-vector space is given by  $s'(x, \vec{f}) = x$ , the target map is given by  $t'(x, \vec{f}) = x + t(\vec{f})$ , and the identity-assigning map is given by  $i'(x) = (x, 0)$ . We thus can define an isomorphism  $\alpha_V: V' \rightarrow V$  by setting

$$\begin{aligned} (\alpha_V)_0(x) &= x, \\ (\alpha_V)_1(x, \vec{f}) &= i(x) + \vec{f}. \end{aligned}$$

It is easy to check that  $\alpha_V$  is a linear functor. It is an isomorphism thanks to the fact, shown in the proof of Lemma 6, that every morphism in  $V$  can be uniquely written as  $i(x) + \vec{f}$  where  $x$  is an object and  $\vec{f} \in \ker(s)$ .

To show that  $\alpha$  is indeed a natural transformation, we must show that given 2-vector spaces  $V, W$  and a linear functor  $F: V \rightarrow W$ , the following diagram commutes:

$$\begin{array}{ccc} ST(V) & \xrightarrow{ST(F)} & ST(W) \\ \alpha_V \downarrow & & \downarrow \alpha_W \\ V & \xrightarrow{F} & W \end{array}$$

where  $ST(F)$  is the linear functor with

$$ST(F)_0 = F_0 \quad \text{and} \quad ST(F)_1 = F_0 \oplus F_1|_{\ker(s)}.$$

This amounts to a straightforward computation.

To construct  $\beta$ , consider a 2-term chain complex,  $C$ , given by

$$C_1 \xrightarrow{d} C_0.$$



Then  $T(C)$  is the 2-vector space with the space  $C_0$  of objects, the space  $C_0 \oplus C_1$  of morphisms, together with the source and target maps  $s: (x, \vec{f}) \mapsto x$ ,  $t: (x, \vec{f}) \mapsto x + d\vec{f}$  and the identity-assigning map  $i: x \mapsto (x, 0)$ . Applying the functor  $S$  to this 2-vector space we obtain a 2-term chain complex  $C'$  given by:

$$\ker(s) \xrightarrow{t|_{\ker(s)}} C_0.$$

Since  $\ker(s) = \{(x, \vec{f}) | x = 0\} \subseteq C_0 \oplus C_1$ , there is an obvious isomorphism  $\ker(s) \cong C_1$ . Using this we obtain an isomorphism  $\beta_C: C' \rightarrow C$  given by:

$$\begin{array}{ccc} \ker(s) & \xrightarrow{t|_{\ker(s)}} & C_0 \\ \downarrow \sim & & \downarrow 1 \\ C_1 & \xrightarrow{d} & C_0 \end{array}$$

where the square commutes because of how we have defined  $t$ .

To show that  $\beta$  is indeed a natural transformation, we must show that given 2-term chain complexes  $C, D$  and a chain map  $\phi: C \rightarrow D$ , the following diagram commutes:

$$\begin{array}{ccc} TS(C) & \xrightarrow{TS(\phi)} & TS(D) \\ \downarrow \beta_C & & \downarrow \beta_D \\ C & \xrightarrow{\phi} & D \end{array}$$

where  $TS(\phi)$  is the chain map with

$$TS(\phi)_0 = \phi_0 \quad \text{and} \quad TS(\phi)_1 = (\phi_0 \oplus \phi_1)|_{C_1} = \phi_1.$$

This amounts to a straightforward computation.  $\square$

As mentioned in the introduction to this chapter, the idea behind Proposition 8 goes back at least to Grothendieck [G], who showed that groupoids in the category of abelian groups are equivalent to 2-term chain complexes of abelian groups. There are many elaborations of this idea, some of which we will mention later, but for now the only one we really *need* involves making  $2\text{Vect}$  and  $2\text{Term}$  into 2-categories and showing that they are 2-equivalent as 2-categories. To do this, we require the notion of a ‘linear natural transformation’ between linear functors. This will correspond to a chain homotopy between chain maps.

**Definition 9.** *Given two linear functors  $F, G: V \rightarrow W$  between 2-vector spaces, a **linear natural transformation**  $\alpha: F \Rightarrow G$  is a natural transformation in  $\text{Vect}$ .*

**Definition 10.** We define **2Vect** to be  $\mathbf{VectCat}$ , or in other words, the 2-category of 2-vector spaces, linear functors and linear natural transformations.

Recall that in general, given two chain maps  $\phi, \psi: C \rightarrow C'$ , a **chain homotopy**  $\tau: \phi \Rightarrow \psi$  is a family of linear maps  $\tau: C_p \rightarrow C'_{p+1}$  such that  $\tau_p d'_{p+1} + d_p \tau_{p-1} = \psi_p - \phi_p$  for all  $p$ . In the case of 2-term chain complexes, a chain homotopy amounts to a map  $\tau: C_0 \rightarrow C'_1$  satisfying  $\tau d' = \psi_0 - \phi_0$  and  $d\tau = \psi_1 - \phi_1$ .

**Definition 11.** We define **2Term** to be the 2-category of 2-term chain complexes, chain maps, and chain homotopies.

We will continue to sometimes use **2Term** and **2Vect** to stand for the underlying categories of these (strict) 2-categories. It will be clear by context whether we mean the category or the 2-category.

The next result strengthens Proposition 8.

**Theorem 12.** The 2-category **2Vect** is 2-equivalent to the 2-category **2Term**.

**Proof.** We begin by constructing 2-functors

$$S: \mathbf{2Vect} \rightarrow \mathbf{2Term}$$

and

$$T: \mathbf{2Term} \rightarrow \mathbf{2Vect}.$$

By Proposition 8, we need only to define  $S$  and  $T$  on 2-morphisms. Let  $V$  and  $V'$  be 2-vector spaces,  $F, G: V \rightarrow V'$  linear functors, and  $\theta: F \Rightarrow G$  a linear natural transformation. Then we define the chain homotopy  $S(\theta): S(F) \Rightarrow S(G)$  by

$$S(\theta)(x) = \vec{\theta}_x,$$

using the fact that a 0-chain  $x$  of  $S(V)$  is the same as an object  $x$  of  $V$ .

To show that  $S(\theta)$  really is a chain homotopy we must show that

$$S(\theta)t|_{\ker(s')} = G_0 - F_0$$

and

$$t|_{\ker(s)} S(\theta) = G_1|_{\ker(s)} - F_1|_{\ker(s)}.$$

To demonstrate the first, we use the fact that  $S(\theta)(x) = \vec{\theta}_x$  is the arrow part of the linear transformation  $\theta$ , so

$$\vec{\theta}_x: 0 \rightarrow t(\theta_x) - s(\theta_x) = G_0(x) - F_0(x)$$

which shows that the target of  $S(\theta)$  is  $G_0 - F_0$ , which is the first condition above. Next, consider  $f: 0 \rightarrow x$ , so that  $f \in \ker(s)$ . Then the commutative square law for a linear natural transformation says

$$\vec{\theta}_0 G(f) = F(f) \vec{\theta}_x.$$

To obtain these composites, we add arrow parts, so that we have

$$G_1(f) = F_1(f) + \vec{\theta}_x = F_1(f) + t|_{\ker(s)} S(\theta)$$

which is precisely the second condition we sought. We thus have that  $S(\theta)$  is a chain homotopy.

Conversely, let  $C$  and  $C'$  be 2-term chain complexes,  $\phi, \psi: C \rightarrow C'$  chain maps and  $\tau: \phi \Rightarrow \psi$  a chain homotopy. Then we define the linear natural transformation  $T(\tau): T(\phi) \Rightarrow T(\psi)$  by

$$T(\tau)(x) = (\phi_0(x), \tau(x)),$$

where we use the description of a morphism in  $S(C')$  as a pair consisting of its source and its arrow part, which is a 1-chain in  $C'$ .

To show that  $T(\tau)$  is really a linear natural transformation, we must show that the three diagrams of Definition 3 commute. We show that the target is correct; showing the source is correct is easier. To show that this diagram

$$\begin{array}{ccc} C_0 & & \\ \downarrow T(\tau) & \searrow \psi_0 & \\ C'_0 \oplus C'_1 & \xrightarrow{t} & C'_0 \end{array}$$

commutes, we observe that

$$t(T(\tau(x))) = t(\phi_0(x), \tau(x)) = \phi_0(x) + d'(\tau(x)) = \phi_0(x) + \psi_0(x) - \phi_0(x) = \psi_0(x)$$

where the third equality follows from the first of the two properties of a chain homotopy following Definition 10. It remains to demonstrate that the commutative square law holds by showing that the following diagram commutes:

$$\begin{array}{ccc} C_0 \oplus C_1 & \xrightarrow{\Delta(sT(\tau) \times \psi)} & (C'_0 \oplus C'_1) \otimes (C'_0 \oplus C'_1) \\ \downarrow \Delta(\phi \times tT(\tau)) & & \downarrow \circ' \\ (C'_0 \oplus C'_1) \otimes (C'_0 \oplus C'_1) & \xrightarrow{\circ'} & C'_0 \oplus C'_1 \end{array}$$

Sending  $(x, \vec{f}) \in C_0 \oplus C_1$  along the top-right path of the above diagram produces

$$(\phi_0(x), \tau(x)) \circ' (\psi_0(x), \psi_1(\vec{f})) = (\phi_0(x), \tau(x) + \psi_1(\vec{f}))$$

whereas sending it along the left-bottom path produces

$$\begin{aligned} (\phi_0(x), \phi_1(\vec{f})) \circ' (\phi_0(x + d(\vec{f})), \tau(x + d(\vec{f}))) &= (\phi_0(x), \phi_1(\vec{f}) + \tau(x + d(\vec{f}))) \\ &= (\phi_0(x), \phi_1(\vec{f}) + \tau(x) + \tau(d(\vec{f}))) \\ &= (\phi_0(x), \phi_1(\vec{f}) + \tau(x) + \psi_1(\vec{f}) - \phi_1(\vec{f})) \end{aligned}$$

where the final equality follows from the second of the two properties of a chain homotopy. Thus, we have that the commutative square law holds.

We leave it to the reader to check that the natural isomorphisms  $\alpha: ST \Rightarrow 1_{2\text{Vect}}$  and  $\beta: TS \Rightarrow 1_{2\text{Term}}$  defined in the proof of Proposition 8 extend to this 2-categorical context.  $\square$

We conclude this section with a little categorified linear algebra. We consider the direct sum and tensor product of 2-vector spaces.

**Proposition 13.** *Given 2-vector spaces  $V = (V_0, V_1, s, t, i, \circ)$  and  $V' = (V'_0, V'_1, s', t', i', \circ')$ , there is a 2-vector space  $V \oplus V'$  having:*

- $V_0 \oplus V'_0$  as its vector space of objects,
- $V_1 \oplus V'_1$  as its vector space of morphisms,
- $s \oplus s'$  as its source map,
- $t \oplus t'$  as its target map,
- $i \oplus i'$  as its identity-assigning map, and
- $\circ \oplus \circ'$  as its composition map.

**Proof.** The proof amounts to a routine verification that the diagrams in Definition 1 commute.  $\square$

**Proposition 14.** *Given 2-vector spaces  $V = (V_0, V_1, s, t, i, \circ)$  and  $V' = (V'_0, V'_1, s', t', i', \circ')$ , there is a 2-vector space  $V \otimes V'$  having:*

- $V_0 \otimes V'_0$  as its vector space of objects,
- $V_1 \otimes V'_1$  as its vector space of morphisms,
- $s \otimes s'$  as its source map,
- $t \otimes t'$  as its target map,
- $i \otimes i'$  as its identity-assigning map, and
- $\circ \otimes \circ'$  as its composition map.

**Proof.** Again, the proof is a routine verification.  $\square$

We now check the correctness of the above definitions by showing the universal properties of the direct sum and tensor product. These universal properties only require the category structure of  $2\text{Vect}$ , not its 2-category structure, since the necessary diagrams commute ‘on the nose’ rather than merely up to a 2-isomorphism, and uniqueness holds up to isomorphism, not just up to equivalence. The direct sum is what category theorists call a ‘biproduct’: both a product and coproduct, in a compatible way [M]:

**Proposition 15.** *The direct sum  $V \oplus V'$  is the biproduct of the 2-vector spaces  $V$  and  $V'$ , with the obvious inclusions*

$$i: V \rightarrow V \oplus V', \quad i': V' \rightarrow V \oplus V'$$

*and projections*

$$p: V \oplus V' \rightarrow V, \quad p': V \oplus V' \rightarrow V'.$$

**Proof.** Showing that  $V \oplus V'$  is a biproduct requires verifying that the following hold

$$ip = 1_V \quad i'p' = 1_{V'} \quad \text{and} \quad pi + p'i' = 1_{V \oplus V'}$$

which is clear.  $\square$

Since the direct sum  $V \oplus V'$  is a product in the categorical sense, we may also denote it by  $V \times V'$ , as we do now in defining a ‘bilinear functor’, which is used in stating the universal property of the tensor product:

**Definition 16.** *Let  $V, V'$ , and  $W$  be 2-vector spaces. A **bilinear functor**  $F: V \times V' \rightarrow W$  is a functor such that the underlying map on objects*

$$F_0: V_0 \times V'_0 \rightarrow W_0$$

*and the underlying map on morphisms*

$$F_1: V_1 \times V'_1 \rightarrow W_1$$

*are bilinear.*

**Proposition 17.** *Let  $V, V'$ , and  $W$  be 2-vector spaces. Given a bilinear functor  $F: V \times V' \rightarrow W$  there exists a unique linear functor  $\tilde{F}: V \otimes V' \rightarrow W$  such that*

$$\begin{array}{ccc} V \times V' & \xrightarrow{F} & W \\ \downarrow i & \nearrow \tilde{F} & \\ V \otimes V' & & \end{array}$$

*commutes, where  $i: V \times V' \rightarrow V \otimes V'$  is given by  $(v, w) \mapsto v \otimes w$  for  $(v, w) \in (V \times V')_0$  and  $(f, g) \mapsto f \otimes g$  for  $(f, g) \in (V \times V')_1$ .*

**Proof.** The existence and uniqueness of  $\tilde{F}_0: (V \otimes V')_0 \rightarrow W_0$  and  $\tilde{F}_1: (V \otimes V')_1 \rightarrow W_1$  follow from the universal property of the tensor product of vector spaces, and it is then straightforward to check that  $\tilde{F}$  is a linear functor.  $\square$

We can also form the tensor product of linear functors. Given linear functors  $F: V \rightarrow V'$  and  $G: W \rightarrow W'$ , we define  $F \otimes G: V \otimes V' \rightarrow W \otimes W'$  by setting:

$$\begin{aligned} (F \otimes G)_0 &= F_0 \otimes G_0, \\ (F \otimes G)_1 &= F_1 \otimes G_1. \end{aligned}$$

Furthermore, there is an ‘identity object’ for the tensor product of 2-vector spaces. In  $\mathbf{Vect}$ , the ground field  $k$  acts as the identity for tensor product: there are canonical isomorphisms  $k \otimes V \cong V$  and  $V \otimes k \cong V$ . For 2-vector spaces, a categorified version of the ground field plays this role:

**Proposition 18.** *There exists a unique 2-vector space  $K$ , the **categorified ground field**, with  $K_0 = K_1 = k$  and  $s, t, i = 1_k$ .*

**Proof.** Lemma 6 implies that there is a unique way to define composition in  $K$  making it into a 2-vector space. In fact, every morphism in  $K$  is an identity morphism.  $\square$

**Proposition 19.** *Given any 2-vector space  $V$ , there is an isomorphism  $\ell_V: K \otimes V \rightarrow V$ , which is defined on objects by  $a \otimes v \mapsto av$  and on morphisms by  $a \otimes f \mapsto af$ . There is also an isomorphism  $r_V: V \otimes K \rightarrow V$ , defined similarly.*

**Proof.** This is straightforward.  $\square$

The functors  $\ell_V$  and  $r_V$  are a categorified version of left and right multiplication by scalars. Our 2-vector spaces also have a categorified version of addition, namely a linear functor

$$+: V \oplus V \rightarrow V$$

mapping any pair  $(x, y)$  of objects or morphisms to  $x + y$ . Combining this with scalar multiplication by the object  $-1 \in K$ , we obtain another linear functor

$$-: V \oplus V \rightarrow V$$

mapping  $(x, y)$  to  $x - y$ . This is the sense in which our 2-vector spaces are equipped with a categorified version of subtraction. All the usual rules governing addition of vectors, subtraction of vectors, and scalar multiplication hold ‘on the nose’ as equations.

One can show that with the above tensor product, the category  $2\mathbf{Vect}$  becomes a symmetric monoidal category. One can go further and make the 2-category version of  $2\mathbf{Vect}$  into a symmetric monoidal 2-category [DS], but we will not need this here. Now that we have a definition of 2-vector space and some basic tools of categorified linear algebra we may proceed to the main focus of this chapter: the definition of a categorified Lie algebra.

## 2.3 Semistrict Lie 2-algebras

### 2.3.1 Definitions

We now introduce the concept of a ‘Lie 2-algebra’, which blends together the notion of a Lie algebra with that of a category. As mentioned previously, a Lie 2-algebra is a 2-vector space equipped with a bracket *functor*, which satisfies the Jacobi identity *up to a natural isomorphism*, the ‘Jacobiator’. Then we require that the Jacobiator satisfy a new coherence law of its own, the ‘Jacobiator identity’. We shall assume the bracket is bilinear in the sense of Definition 16, and also skew-symmetric:

**Definition 20.** Let  $V$  and  $W$  be 2-vector spaces. A bilinear functor  $F: V \times V \rightarrow W$  is **skew-symmetric** if  $F(x, y) = -F(y, x)$  whenever  $(x, y)$  is an object or morphism of  $V \times V$ . If this is the case we also say the corresponding linear functor  $\tilde{F}: V \otimes V \rightarrow W$  is *skew-symmetric*.

We shall also assume that the Jacobiator is trilinear and completely antisymmetric:

**Definition 21.** Let  $V$  and  $W$  be 2-vector spaces. A functor  $F: V^n \rightarrow W$  is  **$n$ -linear** if  $F(x_1, \dots, x_n)$  is linear in each argument, where  $(x_1, \dots, x_n)$  is an object or morphism of  $V^n$ . Given  $n$ -linear functors  $F, G: V^n \rightarrow W$ , a natural transformation  $\theta: F \Rightarrow G$  is  **$n$ -linear** if  $\theta_{x_1, \dots, x_n}$  depends linearly on each object  $x_i$ , and **completely antisymmetric** if the arrow part of  $\theta_{x_1, \dots, x_n}$  is completely antisymmetric under permutations of the objects.

Since we do not weaken the bilinearity or skew-symmetry of the bracket, we call the resulting sort of Lie 2-algebra ‘semistrict’:

**Definition 22.** A **semistrict Lie 2-algebra** consists of:

- a 2-vector space  $L$

equipped with

- a skew-symmetric bilinear functor, the **bracket**,  $[\cdot, \cdot]: L \times L \rightarrow L$
- a completely antisymmetric trilinear natural isomorphism, the **Jacobiator**,

$$J_{x,y,z}: [[x, y], z] \rightarrow [x, [y, z]] + [[x, z], y],$$

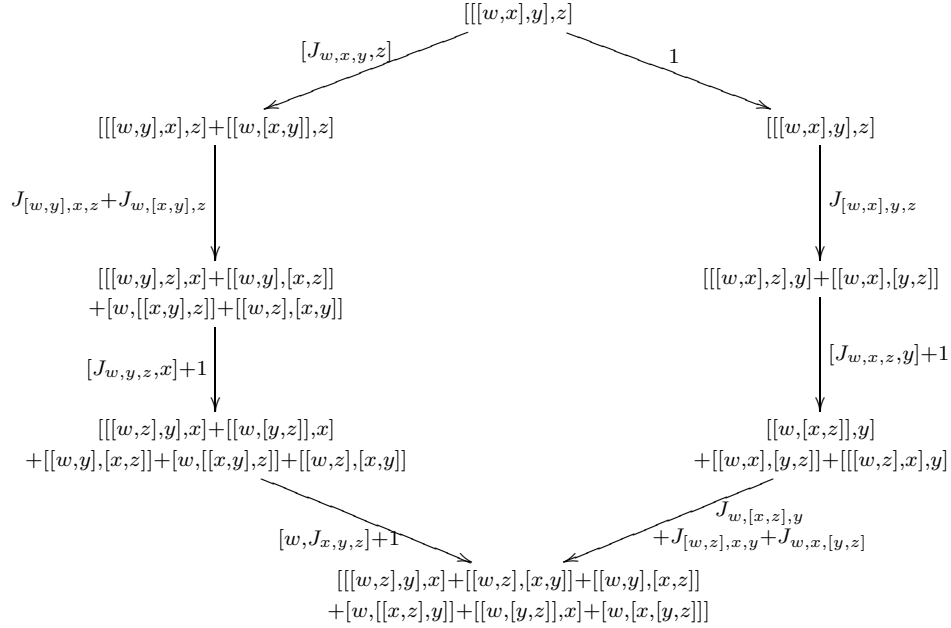
that is required to satisfy

- the **Jacobiator identity**:

$$\begin{aligned} & J_{[w,x],y,z}([J_{w,x,z}, y] + 1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) = \\ & [J_{w,x,y}, z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1) \end{aligned}$$

for all  $w, x, y, z \in L_0$ , where the identity morphisms are chosen to ensure that the target of each morphism being composed is the source of the next one. (There is only one choice of identity morphism that can be added to each term to make the composite well-defined.)

The Jacobiator identity looks quite intimidating at first. But if we draw it as a commutative diagram, we see that it relates two ways of using the Jacobiator to rebracket the expression  $[[[w, x], y], z]$ :



Here the identity morphisms come from terms on which we are not performing any manipulation.

Typically, it is unusual to label an edge of a commutative diagram solely by an identity morphism, as we do for the first arrow on the right. We include it here because in Section 3.3 we show that the Jacobiator identity is really just a disguised version of the ‘Zamolodchikov tetrahedron equation’, which plays an important role in the theory of higher-dimensional knots and braided monoidal 2-categories [BN, CS, C, DS, KV]. The Zamolodchikov tetrahedron equation says that two 2-morphisms are equal, each of which is the vertical composite of four factors. However, when we translate this equation into the language of Lie 2-algebras, one of these factors is an identity 2-morphism.

Until Section 3.3 of the following chapter, the term ‘Lie 2-algebra’ will always refer to a semistrict one as defined above. We continue by setting up a 2-category of these Lie 2-algebras. A homomorphism between Lie 2-algebras should preserve both the 2-vector space structure and the bracket. However, we shall require that it preserve the bracket only *up to isomorphism* — or more precisely, up to a natural isomorphism satisfying a suitable coherence law. Thus, we make the following definition.

**Definition 23.** *Given Lie 2-algebras  $L$  and  $L'$ , a **homomorphism**  $F: L \rightarrow L'$  consists of:*

- *A linear functor  $F$  from the underlying 2-vector space of  $L$  to that of  $L'$ , and*
- *a skew-symmetric bilinear natural transformation*

$$F_2(x, y): [F_0(x), F_0(y)] \rightarrow F_0[x, y]$$



such that the following diagram commutes:

$$\begin{array}{ccc}
[F_0(x), [F_0(y), F_0(z)]] & \xrightarrow{J_{F_0(x), F_0(y), F_0(z)}} & [[F_0(x), F_0(y)], F_0(z)] + [F_0(y), [F_0(x), F_0(z)]] \\
\downarrow [1, F_2] & & \downarrow [F_2, 1] + [1, F_2] \\
[F_0(x), F_0[y, z]] & & [F_0[x, y], F_0(z)] + [F_0(y), F_0[x, z]] \\
\downarrow F_2 & & \downarrow F_2 + F_2 \\
F_0[x, [y, z]] & \xrightarrow{F_1(J_{x, y, z})} & F_0[[x, y], z] + F_0[y, [x, z]]
\end{array}$$

Here and elsewhere we omit the arguments of natural transformations such as  $F_2$  and  $G_2$  when these are obvious from context.

We also have ‘2-homomorphisms’ between homomorphisms, which are linear natural transformations with an extra property:

**Definition 24.** Let  $F, G: L \rightarrow L'$  be Lie 2-algebra homomorphisms. A **2-homomorphism**  $\theta: F \Rightarrow G$  is a linear natural transformation from  $F$  to  $G$  such that the following diagram commutes:

$$\begin{array}{ccc}
[F_0(x), F_0(y)] & \xrightarrow{F_2} & F_0[x, y] \\
\downarrow [\theta_x, \theta_y] & & \downarrow \theta_{[x, y]} \\
[G_0(x), G_0(y)] & \xrightarrow{G_2} & G_0[x, y]
\end{array}$$

Definitions 23 and 24 are closely modelled after the usual definitions of ‘monoidal functor’ and ‘monoidal natural transformation’ [M].

Next we introduce composition and identities for homomorphisms and 2-homomorphisms. The composite of a pair of Lie 2-algebra homomorphisms  $F: L \rightarrow L'$  and  $G: L' \rightarrow L''$  is given by letting the functor  $FG: L \rightarrow L''$  be the usual composite of  $F$  and  $G$ :

$$L \xrightarrow{F} L' \xrightarrow{G} L''$$

while letting  $(FG)_2$  be defined as the following composite:

$$\begin{array}{ccc}
[(FG)_0(x), (FG)_0(y)] & \xrightarrow{(FG)_2} & (FG)_0[x, y] \\
\downarrow G_2 & \nearrow F_2 \circ G & \\
G_0[F_0(x), F_0(y)] & & 
\end{array}$$

where  $F_2 \circ G$  is the result of whiskering the functor  $G$  by the natural transformation  $F_2$ . The identity homomorphism  $1_L: L \rightarrow L$  has the identity functor as its underlying functor, together with an identity natural transformation as  $(1_L)_2$ . Since 2-homomorphisms are just natural transformations with an extra property, we vertically and horizontally compose these the usual way, and an identity 2-homomorphism is just an identity natural transformation. We obtain:

**Proposition 25.** *There is a strict 2-category **Lie2Alg** with semistrict Lie 2-algebras as objects, homomorphisms between these as morphisms, and 2-homomorphisms between those as 2-morphisms, with composition and identities defined as above.*

**Proof.** We leave it to the reader to check the details, including that the composite of homomorphisms is a homomorphism, this composition is associative, and the vertical and horizontal composites of 2-homomorphisms are again 2-homomorphisms.  $\square$

Finally, note that there is a forgetful 2-functor from **Lie2Alg** to **2Vect**, which is analogous to the forgetful functor from **LieAlg** to **Vect**.

We continue by exhibiting the correlation between our semistrict Lie 2-algebras and special versions of Stasheff's  $L_\infty$ -algebras.

### 2.3.2 $L_\infty$ -algebras

An  $L_\infty$ -algebra is a generalization, or homotopy version, of a Lie algebra. More specifically, an  $L_\infty$ -algebra is a chain complex equipped with a bilinear skew-symmetric bracket operation that satisfies the Jacobi identity up to an infinite tower of chain homotopies, thereby blending together the notion of a Lie algebra with that of a chain complex. Such structures are also called are ‘strongly homotopy Lie algebras’ or ‘sh Lie algebras’ for short. Though they had existed in the literature beforehand, they made their first notable appearance in a 1985 paper on deformation theory by Schlessinger and Stasheff [SS]. Since then, they have been systematically explored and applied in a number of other contexts [LM, LS, Mar, P].

Since 2-vector spaces are equivalent to 2-term chain complexes, as described in Section 2.2, it should not be surprising that  $L_\infty$ -algebras are related to the categorified Lie algebras we defined in the previous section. Indeed, after recalling the definition of an  $L_\infty$ -algebra we prove that the 2-category of semistrict Lie 2-algebras is equivalent to a 2-category of ‘2-term’  $L_\infty$ -algebras: that is, those having a zero-dimensional space of  $j$ -chains for  $j > 1$ .

Henceforth, all algebraic objects mentioned are considered over a fixed field  $k$  of characteristic other than 2. We make consistent use of the usual sign convention when dealing with graded objects. That is, whenever we interchange something of degree  $p$  with something of degree  $q$ , we introduce a sign of  $(-1)^{pq}$ . The following conventions regarding graded vector spaces, permutations, unshuffles, etc., follow those of Lada and Markl [LM].

For graded indeterminates  $x_1, \dots, x_n$  and a permutation  $\sigma \in S_n$  we define the **Koszul sign**  $\epsilon(\sigma) = \epsilon(\sigma; x_1, \dots, x_n)$  by

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \dots, x_n) \cdot x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

which must be satisfied in the free graded-commutative algebra on  $x_1, \dots, x_n$ . This is nothing more than a formalization of what has already been said above. Furthermore, we define

$$\chi(\sigma) = \chi(\sigma; x_1, \dots, x_n) := \text{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \dots, x_n).$$

Thus,  $\chi(\sigma)$  takes into account the sign of the permutation in  $S_n$  and the sign obtained from iteration of the basic convention.

If  $n$  is a natural number and  $1 \leq j \leq n-1$  we say that  $\sigma \in S_n$  is an  $(j, n-j)$ -**unshuffle** if

$$\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(j) \quad \text{and} \quad \sigma(j+1) \leq \sigma(j+2) \leq \cdots \leq \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses. A *shuffle* of two ordered sets (such as a deck of cards) is a permutation of the ordered union preserving the order of each of the given subsets. An *unshuffle* reverses this process. A simple example should clear up any confusion:

**Example 26.** When  $n = 3$ , the  $(1, 2)$ -unshuffles in  $S_3$  are:

$$\text{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The following definition of an  $L_\infty$ -structure was formulated by Stasheff in 1985, see [SS]. This definition will play an important role in what will follow.

**Definition 27.** An  **$L_\infty$ -algebra** is a graded vector space  $V$  equipped with a system  $\{l_k | 1 \leq k < \infty\}$  of linear maps  $l_k: V^{\otimes k} \rightarrow V$  with  $\deg(l_k) = k - 2$  which are totally antisymmetric in the sense that

$$l_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \dots, x_n) \tag{2.1}$$

for all  $\sigma \in S_n$  and  $x_1, \dots, x_n \in V$ , and, moreover, the following generalized form of the Jacobi identity holds for  $0 \leq n < \infty$ :

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma) (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0, \tag{2.2}$$

where the summation is taken over all  $(i, n-i)$ -unshuffles with  $i \geq 1$ .

While it appears complicated at first, this definition truly does combine the important aspects of Lie algebras and chain complexes. The map  $l_1$  makes  $V$  into a chain complex, since this map has degree  $-1$  and equation (2.2) says its square is zero. Moreover, the map  $l_2$  resembles a Lie bracket, since it is skew-symmetric in the graded sense by equation (2.1). In what follows, we usually denote  $l_1(x)$  as  $dx$  and  $l_2(x, y)$  as  $[x, y]$ . The higher  $l_k$  maps are related to the Jacobiator, the Jacobiator identity, and the higher coherence laws that would appear upon further categorification of the Lie algebra concept.

To make this more precise, let us refer to an  $L_\infty$ -algebra  $V$  with  $V_n = 0$  for  $n \geq k$  as a  **$k$ -term  $L_\infty$ -algebra**. Note that a 1-term  $L_\infty$ -algebra is simply an ordinary Lie algebra, where  $l_3 = 0$  gives the Jacobi identity. However, in a 2-term  $L_\infty$ -algebra, we no longer have a trivial  $l_3$  map. Instead, equation (2.2) says that the Jacobi identity for the 0-chains  $x, y, z$  holds up to a term of the form  $dl_3(x, y, z)$ . We do, however, have  $l_4 = 0$ , which provides us with the coherence law that  $l_3$  must satisfy.

Since we will be making frequent use of these 2-term  $L_\infty$ -algebras, it will be advantageous to keep track of their ingredients.

**Lemma 28.** *A 2-term  $L_\infty$ -algebra,  $V$ , consists of the following data:*

- two vector spaces  $V_0$  and  $V_1$  together with a linear map  $d: V_1 \rightarrow V_0$ ,
- a bilinear map  $l_2: V_i \times V_j \rightarrow V_{i+j}$ , where  $0 \leq i + j \leq 1$ ,  
which we denote more suggestively as  $[\cdot, \cdot]$ ,
- a trilinear map  $l_3: V_0 \times V_0 \times V \rightarrow V_1$ .

*These maps satisfy:*

- (a)  $[x, y] = -[y, x]$ ,
- (b)  $[x, h] = -[h, x]$ ,
- (c)  $[h, k] = 0$ ,
- (d)  $l_3(x, y, z)$  is totally antisymmetric in the arguments  $x, y, z$ ,
- (e)  $d([x, h]) = [x, dh]$ ,
- (f)  $[dh, k] = [h, dk]$ ,
- (g)  $d(l_3(x, y, z)) = -[[x, y], z] + [[x, z], y] + [x, [y, z]]$ ,
- (h)  $l_3(dh, x, y) = -[[x, y], h] + [[x, h], y] + [x, [y, h]]$ ,
- (i)  $[l_3(w, x, y), z] + [l_3(w, y, z), x] + l_3([w, y], x, z) + l_3([x, z], w, y) =$   
 $[l_3(w, x, z), y] + [l_3(x, y, z), w] + l_3([w, x], y, z) + l_3([w, z], x, y) + l_3([x, y], w, z) + l_3([y, z], w, x),$

for all  $w, x, y, z \in V_0$  and  $h, k \in V_1$ .

Proof. Note that (a) – (d) hold by equation (2.1) of Definition 27 while (e) – (i) follow from (2.2).  $\square$

We notice that (a) and (b) are the usual skew-symmetric properties satisfied by the bracket in a Lie algebra; (c) arises simply because there are no 2-chains. Properties (e) and (f) tell us how the differential and bracket interact, while condition (g) says that the Jacobi identity no longer holds on the nose, but up to chain homotopy. We will use (g) to define the Jacobiator in the Lie 2-algebra corresponding to a 2-term  $L_\infty$ -algebra. Equation (h) will give the naturality of the Jacobiator. Similarly, (i) will give the Jacobiator identity.

We continue by defining homomorphisms between 2-term  $L_\infty$ -algebras:

**Definition 29.** Let  $V$  and  $V'$  be 2-term  $L_\infty$ -algebras. An  **$L_\infty$ -homomorphism**  $\phi: V \rightarrow V'$  consists of:

- a chain map  $\phi: V \rightarrow V'$  (which consists of linear maps  $\phi_0: V_0 \rightarrow V'_0$  and  $\phi_1: V_1 \rightarrow V'_1$  preserving the differential),
- a skew-symmetric bilinear map  $\phi_2: V_0 \times V_0 \rightarrow V'_1$ ,

such that the following equations hold for all  $x, y, z \in V_0$ ,  $h, k \in V_1$ :

- $d\phi_2(x, y) = \phi_0[x, y] - [\phi_0(x), \phi_0(y)]$
- $\phi_2(d(h), d(k)) = \phi_1[h, k] - [\phi_1(h), \phi_1(k)]$
- $\phi_1(l_3(x, y, z)) + \phi_2(x, [y, z]) + [\phi_0(x), \phi_2(y, z)] = l_3(\phi_0(x), \phi_0(y), \phi_0(z)) + \phi_2([x, y], z) + \phi_2(y, [x, z]) + [\phi_2(x, y), \phi_0(z)] + [\phi_0(y), \phi_2(x, z)]$

The reader should note the similarity between this definition and that of homomorphisms between Lie 2-algebras (Definition 23). In particular, the first two equations say that  $\phi_2$  defines a chain homotopy from  $[\phi(\cdot), \phi(\cdot)]$  to  $\phi[\cdot, \cdot]$ , where these are regarded as chain maps from  $V \otimes V$  to  $V'$ . The third equation in the above definition is just a chain complex version of the commutative square in Definition 23.

To make 2-term  $L_\infty$ -algebras and  $L_\infty$ -homomorphisms between them into a category, we must describe composition and identities. We compose a pair of  $L_\infty$ -homomorphisms  $\phi: V \rightarrow V'$  and  $\psi: V' \rightarrow V''$  by letting the chain map  $\phi\psi: V \rightarrow V''$  be the usual composite:

$$V \xrightarrow{\phi} V' \xrightarrow{\psi} V''$$

while defining  $(\phi\psi)_2$  as follows:

$$(\phi\psi)_2(x, y) = \psi_2(\phi_0(x), \phi_0(y)) + \psi_0(\phi_2(x, y)).$$

This is just a chain complex version of how we compose homomorphisms between Lie 2-algebras. The identity homomorphism  $1_V: V \rightarrow V$  has the identity chain map as its underlying map, together with  $(1_V)_2 = 0$ .

With these definitions, we obtain:

**Proposition 30.** *There is a category  $2\text{Term}L_\infty$  with 2-term  $L_\infty$ -algebras as objects and  $L_\infty$ -homomorphisms as morphisms.*

**Proof.** We must show that the identity homomorphism behaves as it should and that composition of  $L_\infty$ -homomorphisms is associative. Obviously, we need only to check these for the skew-symmetric bilinear maps, since the chain maps satisfy these conditions. To check the left unit law, we consider

$$V \xrightarrow{1_V} V \xrightarrow{\phi} V'$$

Then

$$(1_V \phi)_2 = \phi_2(1_{V_0}(x, y)) + \phi_0(0) = \phi_2(x, y)$$

as desired. The proof of the right unit law is similar. To demonstrate associativity, consider

$$V \xrightarrow{\tau} V' \xrightarrow{\phi} V'' \xrightarrow{\psi} V'''$$

On one hand,

$$\begin{aligned} (\tau(\phi\psi))_2(x, y) &= (\phi\psi)_2(\tau_0(x), \tau_0(y)) + (\phi\psi)_0(\tau_2(x, y)) \\ &= \psi_2(\phi_0(\tau_0(x)), \phi_0(\tau_0(y))) + \psi_0(\phi_2(\tau_0(x), \tau_0(y))) + (\phi\psi)_0(\tau_2(x, y)) \end{aligned}$$

while on the other hand,

$$\begin{aligned} ((\tau\phi)\psi)_2(x, y) &= \psi_2((\tau\phi)_0(x), (\tau\phi)_0(y)) + \psi_0((\tau\phi)_2(x, y)) \\ &= \psi_2((\tau\phi)_0(x), (\tau\phi)_0(y)) + \psi_0(\phi_2(\tau_0(x), \tau_0(y)) + \phi_0(\tau_2(x, y))). \end{aligned}$$

Recalling our conventions regarding composition, these two expressions are identical.  $\square$

Next we establish the equivalence between the category of Lie 2-algebras and that of 2-term  $L_\infty$ -algebras. This result is based on the equivalence between 2-vector spaces and 2-term chain complexes described in Proposition 8.

**Theorem 31.** *The categories  $\text{Lie2Alg}$  and  $2\text{Term}L_\infty$  are equivalent.*

**Proof.** First we sketch how to construct a functor  $T: 2\text{Term}L_\infty \rightarrow \text{Lie2Alg}$ . Given a 2-term  $L_\infty$ -algebra  $V$  we construct the Lie 2-algebra  $L = T(V)$  as follows.

We construct the underlying 2-vector space of  $L$  as in the proof of Proposition 8. Thus  $L$  has vector spaces of objects and morphisms

$$\begin{aligned} L_0 &= V_0, \\ L_1 &= V_0 \oplus V_1, \end{aligned}$$

and we denote a morphism  $f: x \rightarrow y$  in  $L_1$  by  $f = (x, \vec{f})$  where  $x \in V_0$  is the source of  $f$  and  $\vec{f} \in V_1$  is its arrow part. The source, target, and identity-assigning maps in  $L$  are given by

$$\begin{aligned} s(f) &= s(x, \vec{f}) = x, \\ t(f) &= t(x, \vec{f}) = x + d\vec{f}, \\ i(x) &= (x, 0), \end{aligned}$$

and we have  $t(f) - s(f) = d\vec{f}$ . The composite of two morphisms in  $L_1$  is given as in the proof of Lemma 6. That is, given  $f = (x, \vec{f}): x \rightarrow y$ , and  $g = (y, \vec{g}): y \rightarrow z$ , we have

$$fg := (x, \vec{f} + \vec{g}).$$

We continue by equipping  $L = T(V)$  with the additional structure which makes it a Lie 2-algebra. First, we use the degree-zero chain map  $l_2: V \otimes V \rightarrow V$  to define the bracket functor  $[\cdot, \cdot]: L \times L \rightarrow L$ . For a pair of objects  $x, y \in L_0$  we define  $[x, y] = l_2(x, y)$ , where we use the ' $l_2$ ' notation in the  $L_\infty$ -algebra  $V$  to avoid confusion with the bracket in  $L$ . The bracket functor is skew-symmetric and bilinear on objects since  $l_2$  is. This is not sufficient, however. It remains to define the bracket functor on pairs of morphisms.

We begin by defining the bracket on pairs of morphisms where one morphism is an identity. We do this as follows: given a morphism  $f = (x, \vec{f}): x \rightarrow y$  in  $L_1$  and an object  $z \in L_0$ , we define

$$\begin{aligned} [1_z, f] &:= (l_2(z, x), l_2(z, \vec{f})), \\ [f, 1_z] &:= (l_2(x, z), l_2(\vec{f}, z)). \end{aligned}$$

Clearly these morphisms have the desired sources; we now verify that they also have the desired targets. Using the fact that  $t(f) = s(f) + d\vec{f}$  for any morphism  $f \in L_1$ , we see that:

$$\begin{aligned} t[1_z, f] &= s[1_z, f] + dl_2(z, \vec{f}) \\ &= l_2(z, x) + l_2(z, d\vec{f}) \quad \text{by (e) of Lemma 28} \\ &= l_2(z, x) + l_2(z, y - x) \\ &= l_2(z, y) \end{aligned}$$

as desired, using the bilinearity of  $l_2$ . Similarly we have:

$$\begin{aligned} t[f, 1_z] &= s[f, 1_z] + dl_2(\vec{f}, z) \\ &= l_2(x, z) + l_2(d\vec{f}, z) \quad \text{by (e) of Lemma 28} \\ &= l_2(x, z) + l_2(y - x, z) \\ &= l_2(y, z) \end{aligned}$$

again using the bilinearity of  $l_2$ .

These definitions together with the desired functoriality of the bracket force us to define the bracket of an arbitrary pair of morphisms  $f: x \rightarrow y$ ,  $g: a \rightarrow b$  as follows:

$$\begin{aligned} [f, g] &= [f1_y, 1_ag] \\ &:= [f, 1_a][1_y, g] \\ &= (l_2(x, a), l_2(\vec{f}, a))(l_2(y, a), l_2(y, \vec{g})) \\ &= (l_2(x, a), l_2(\vec{f}, a) + l_2(y, \vec{g})). \end{aligned}$$

On the other hand, they also force us to define it as:

$$\begin{aligned} [f, g] &= [1_x f, g1_b] \\ &:= [1_x, g][f, 1_b] \\ &= (l_2(x, a), l_2(x, \vec{g}))(l_2(x, b), l_2(\vec{f}, b)) \\ &= (l_2(x, a), l_2(x, \vec{g}) + l_2(\vec{f}, b)). \end{aligned}$$

Luckily these are compatible! We have

$$l_2(\vec{f}, a) + l_2(y, \vec{g}) = l_2(x, \vec{g}) + l_2(\vec{f}, b) \quad (2.3)$$

because the left-hand side minus the right-hand side equals  $l_2(d\vec{f}, \vec{g}) - l_2(\vec{f}, d\vec{g})$ , which vanishes by (f) of Lemma 28.

At this point we relax and use  $[\cdot, \cdot]$  to stand both for the bracket in  $L$  and the  $L_\infty$ -algebra  $V$ . In this new relaxed notation the bracket of morphisms  $f: x \rightarrow y$ ,  $g: a \rightarrow b$  in  $L$  is given by

$$\begin{aligned} [f, g] &= ([x, a], [\vec{f}, a] + [y, \vec{g}]) \\ &= ([x, a], [x, \vec{g}] + [\vec{f}, b]). \end{aligned}$$

The bracket  $[\cdot, \cdot]: L \times L \rightarrow L$  is clearly bilinear on objects. Either of the above formulas shows it is also bilinear on morphisms, since the source, target and arrow part of a morphism depend linearly on the morphism, and the bracket in  $V$  is bilinear. The bracket is also skew-symmetric: this is clear for objects, and can be seen for morphisms if we use *both* the above formulas.

To show that  $[\cdot, \cdot]: L \times L \rightarrow L$  is a functor, we must check that it preserves identities and composition. We first show that  $[1_x, 1_y] = 1_{[x, y]}$ , where  $x, y \in L_0$ . For this we use the fact that identity morphisms are precisely those with vanishing arrow part. Either formula for the bracket of morphisms gives

$$\begin{aligned} [1_x, 1_y] &= ([x, y], 0) \\ &= 1_{[x, y]}. \end{aligned}$$

To show that the bracket preserves composition, consider the morphisms  $f = (x, \vec{f})$ ,  $f' = (y, \vec{f}')$ ,  $g = (a, \vec{g})$ , and  $g' = (b, \vec{g}')$  in  $L_1$ , where  $f: x \rightarrow y$ ,  $f': y \rightarrow z$ ,  $g: a \rightarrow b$ , and  $g': b \rightarrow c$ . We must show

$$[ff', gg'] = [f, g][f', g'].$$

On the one hand, the definitions give

$$[ff', gg'] = ([x, a], [\vec{f}, a] + [\vec{f}', a] + [z, \vec{g}] + [z, \vec{g}']),$$

while on the other, they give

$$[f, g][f', g'] = ([x, a], [\vec{f}, a] + [y, \vec{g}] + [\vec{f}', b] + [z, \vec{g}'])$$

Therefore, it suffices to show that

$$[\vec{f}', a] + [z, \vec{g}] = [y, \vec{g}] + [\vec{f}', b].$$

After a relabelling of variables, this is just equation (2.3).

Next we define the Jacobiator for  $L$  and check its properties. We set

$$J_{x, y, z} := ([x, y], [z], l_3(x, y, z)).$$



Clearly the source of  $J_{x,y,z}$  is  $[[x, y], z]$  as desired, while its target is  $[x, [y, z]] + [[x, z], y]$  by condition (g) of Lemma 28. To show  $J_{x,y,z}$  is natural one can check that is natural in each argument. We check naturality in the third variable, leaving the other two as exercises for the reader. Let  $f: z \rightarrow z'$ . Then,  $J_{x,y,z}$  is natural in  $z$  if the following diagram commutes:

$$\begin{array}{ccc}
[[x, y], z] & \xrightarrow{[[1_x, 1_y], f]} & [[x, y], z'] \\
\downarrow J_{x,y,z} & & \downarrow J_{x,y,z'} \\
[[x, z], y] + [x, [y, z]] & \xrightarrow{[[1_x, f], 1_y] + [1_x, [1_y, f]]} & [[x, z'], y] + [x, [y, z']]
\end{array}$$

Using the formula for the composition and bracket in  $L$  this means that we need

$$([x, y], z], \vec{J}_{x,y,z'} + [[x, y], \vec{f}]) = ([x, y], z], [[x, \vec{f}], y] + [x, [y, \vec{f}]] + \vec{J}_{x,y,z}).$$

Thus, it suffices to show that

$$\vec{J}_{x,y,z'} + [[x, y], \vec{f}] = [[x, \vec{f}], y] + [x, [y, \vec{f}]] + \vec{J}_{x,y,z}.$$

But  $\vec{J}_{x,y,z}$  has been defined as  $l_3(x, y, z)$  (and similarly for  $\vec{J}_{x,y,z'}$ ), so now we are required to show that:

$$l_3(x, y, z') + [[x, y], \vec{f}] = l_3(x, y, z) + [[x, \vec{f}], y] + [x, [y, \vec{f}]],$$

or in other words,

$$[[x, y], \vec{f}] + l_3(x, y, d\vec{f}) = [[x, \vec{f}], y] + [x, [y, \vec{f}]].$$

This holds by condition (h) in Lemma 28 together with the complete antisymmetry of  $l_3$ .

The trilinearity and complete antisymmetry of the Jacobiator follow from the corresponding properties of  $l_3$ . Finally, condition (i) in Lemma 28 gives the Jacobiator identity:

$$\begin{aligned}
& J_{[w,x],y,z}([J_{w,x,z}, y] + 1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) = \\
& [J_{w,x,y}, z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1).
\end{aligned}$$

This completes the construction of a Lie 2-algebra  $T(V)$  from any 2-term  $L_\infty$ -algebra  $V$ . Next we construct a Lie 2-algebra homomorphism  $T(\phi): T(V) \rightarrow T(V')$  from any  $L_\infty$ -homomorphism  $\phi: V \rightarrow V'$  between 2-term  $L_\infty$ -algebras.

Let  $T(V) = L$  and  $T(V') = L'$ . We define the underlying linear functor of  $T(\phi) = F$  as in Proposition 8. To make  $F$  into a Lie 2-algebra homomorphism we must equip it with a skew-symmetric bilinear natural transformation  $F_2$  satisfying the conditions in Definition 23. We do this using the skew-symmetric bilinear map  $\phi_2: V_0 \times V_0 \rightarrow V'_1$ . In terms of its source and arrow parts, we let

$$F_2(x, y) = ([\phi_0(x), \phi_0(y)], \phi_2(x, y)).$$

Computing the target of  $F_2(x, y)$  we have:

$$\begin{aligned}
tF_2(x, y) &= sF_2(x, y) + d\vec{F}_2(x, y) \\
&= [\phi_0(x), \phi_0(y)] + d\phi_2(x, y) \\
&= [\phi_0(x), \phi_0(y)] + \phi_0[x, y] - [\phi_0(x), \phi_0(y)] \\
&= \phi_0[x, y] \\
&= F_0[x, y]
\end{aligned}$$

by the first equation in Definition 29 and the fact that  $F_0 = \phi_0$ . Thus we have  $F_2(x, y): [F_0(x), F_0(y)] \rightarrow F_0[x, y]$ . Notice that  $F_2(x, y)$  is bilinear and skew-symmetric since  $\phi_2$  and the bracket are.  $F_2$  is a natural transformation by Theorem 12 and the fact that  $\phi_2$  is a chain homotopy from  $[\phi(\cdot), \phi(\cdot)]$  to  $\phi([\cdot, \cdot])$ , thought of as chain maps from  $V \otimes V$  to  $V'$ . Finally, the equation in Definition 29 gives the commutative diagram in Definition 23, since the composition of morphisms corresponds to addition of their arrow parts.

We leave it to the reader to check that  $T$  is indeed a functor. Next, we describe how to construct a functor  $S: \text{Lie2Alg} \rightarrow 2\text{Term}L_\infty$ .

Given a Lie 2-algebra  $L$  we construct the 2-term  $L_\infty$ -algebra  $V = S(L)$  as follows. We define:

$$\begin{aligned}
V_0 &= L_0 \\
V_1 &= \ker(s) \subseteq L_1.
\end{aligned}$$

In addition, we define the maps  $l_k$  as follows:

- $l_1 h = t(h)$  for  $h \in V_1 \subseteq L_1$ .
- $l_2(x, y) = [x, y]$  for  $x, y \in V_0 = L_0$ .
- $l_2(x, h) = -l_2(h, x) = [1_x, h]$  for  $x \in V_0 = L_0$  and  $h \in V_1 \subseteq L_1$ .
- $l_2(h, k) = 0$  for  $h, k \in V_1 \subseteq L_1$ .
- $l_3(x, y, z) = \vec{J}_{x, y, z}$  for  $x, y, z \in V_0 = L_0$ .

As usual, we abbreviate  $l_1$  as  $d$  and  $l_2$  as  $[\cdot, \cdot]$ .

With these definitions, conditions (a) and (b) of Lemma 28 follow from the anti-symmetry of the bracket functor. Condition (c) is automatic. Condition (d) follows from the complete antisymmetry of the Jacobiator.

For  $h \in V_1$  and  $x \in V_0$ , the functoriality of  $[\cdot, \cdot]$  gives

$$\begin{aligned}
d([x, h]) &= t([1_x, h]) \\
&= [t(1_x), t(h)] \\
&= [x, dh],
\end{aligned}$$

which is (e) of Lemma 28. To obtain (f), 28, we let  $h: 0 \rightarrow x$  and  $k: 0 \rightarrow y$  be elements of  $V_1$ . We then consider the following square in  $L \times L$ ,

$$\begin{array}{ccccc}
& & 0 & \xrightarrow{h} & x \\
& & \downarrow k & & \downarrow \\
0 & \xrightarrow{(h,1_0)} & (0,0) & \xrightarrow{(h,1_0)} & (x,0) \\
& \downarrow (1_0,k) & \downarrow (1_0,k) & & \downarrow (1_x,k) \\
& & (0,y) & \xrightarrow{(h,1_y)} & (x,y)
\end{array}$$

which commutes by definition of a product category. Since  $[\cdot, \cdot]$  is a functor, it preserves such commutative squares, so that

$$\begin{array}{ccc}
[0, 0] & \xrightarrow{[h, 1_0]} & [x, 0] \\
\downarrow [1_0, k] & & \downarrow [1_x, k] \\
[0, y] & \xrightarrow{[h, 1_y]} & [x, y]
\end{array}$$

commutes. Since  $[h, 1_0]$  and  $[1_0, k]$  are easily seen to be identity morphisms, this implies  $[h, 1_y] = [1_x, k]$ . This means that in  $V$  we have  $[h, y] = [x, k]$ , or, since  $y$  is the target of  $k$  and  $x$  is the target of  $h$ , simply  $[h, dk] = [dh, k]$ , which is (f) of Lemma 28.

Since  $J_{x,y,z}: [[x, y], z] \rightarrow [x, [y, z]] + [[x, z], y]$ , we have

$$\begin{aligned}
d(l_3(x, y, z)) &= t(\vec{J}_{x,y,z}) \\
&= (t - s)(J_{x,y,z}) \\
&= [x, [y, z]] + [[x, z], y] - [[x, y], z],
\end{aligned}$$

which gives (g) of Lemma 28. The naturality of  $J_{x,y,z}$  implies that for any  $f: z \rightarrow z'$ , we must have

$$[[1_x, 1_y], f] \circ J_{x,y,z'} = J_{x,y,z} \circ ([1_x, f], 1_y + [1_x, [1_y, f]]).$$

This implies that in  $V$  we have

$$[[x, y], \vec{f}] + l_3(x, y, z' - z) = [[x, \vec{f}], y] + [x, [y, \vec{f}]],$$

for  $x, y \in V_0$  and  $\vec{f} \in V_1$ , which is (h) of Lemma 28.

Finally, the Jacobiator identity dictates that the arrow part of the Jacobiator,  $l_3$ , satisfies the following equation:

$$\begin{aligned}
&[l_3(w, x, y), z] + [l_3(w, y, z), x] + [l_3([w, y], x, z) + l_3([x, z], w, y) = \\
&[l_3(w, x, z), y] + [l_3(x, y, z), w] + [l_3([w, x], y, z) + l_3([w, z], x, y) + l_3([x, y], w, z) + l_3([y, z], w, x)].
\end{aligned}$$

This is (i) of Lemma 28.

This completes the construction of a 2-term  $L_\infty$ -algebra  $S(L)$  from any Lie 2-algebra  $L$ . Next we construct an  $L_\infty$ -homomorphism  $S(F): S(L) \rightarrow S(L')$  from any Lie 2-algebra homomorphism  $F: L \rightarrow L'$ .

Let  $S(L) = V$  and  $S(L') = V'$ . We define the underlying chain map of  $S(F) = \phi$  as in Proposition 8. To make  $\phi$  into an  $L_\infty$ -homomorphism we must equip it with a skew-symmetric bilinear map  $\phi_2: V_0 \times V_0 \rightarrow V'_1$  satisfying the conditions in Definition 29. To do this we set

$$\phi_2(x, y) = \vec{F}_2(x, y).$$

The bilinearity and skew-symmetry of  $\phi_2$  follow from that of  $F_2$ . Then, since  $\phi_2$  is the arrow part of  $F_2$ ,

$$\begin{aligned} d\phi_2(x, y) &= (t - s)F_2(x, y) \\ &= F_0[x, y] - [F_0(x), F_0(y)] \\ &= \phi_0[x, y] - [\phi_0(x), \phi_0(y)], \end{aligned}$$

by definition of the chain map  $\phi$ . The naturality of  $F_2$  gives the second equation in Definition 29. Finally, since composition of morphisms corresponds to addition of arrow parts, the diagram in Definition 23 gives:

$$\begin{aligned} \phi_1(l_3(x, y, z)) + \phi_2(x, [y, z]) + [\phi_0(x), \phi_2(y, z)] &= l_3(\phi_0(x), \phi_0(y), \phi_0(z)) + \\ \phi_2([x, y], z) + \phi_2(y, [x, z]) + [\phi_2(x, y), \phi_0(z)] + [\phi_0(y), \phi_2(x, z)], \end{aligned}$$

since  $\phi_0 = F_0$ ,  $\phi_1 = F_1$  on elements of  $V_1$ , and the arrow parts of  $J$  and  $F_2$  are  $l_3$  and  $\phi_2$ , respectively.

We leave it to the reader to check that  $S$  is indeed a functor, and to construct natural isomorphisms  $\alpha: ST \Rightarrow 1_{\text{Lie2Alg}}$  and  $\beta: TS \Rightarrow 1_{2\text{Term}L_\infty}$ .  $\square$

The above theorem also has a 2-categorical version. We have defined a 2-category of Lie 2-algebras, but not yet defined a 2-category of 2-term  $L_\infty$ -algebras. For this, we need the following:

**Definition 32.** *Let  $V$  and  $V'$  be 2-term  $L_\infty$ -algebras and let  $\phi, \psi: V \rightarrow V'$  be  $L_\infty$ -homomorphisms. An  **$L_\infty$ -2-homomorphism**  $\tau: \phi \Rightarrow \psi$  is a chain homotopy such that the following equation holds for all  $x, y \in V_0$ :*

$$\bullet \quad \phi_2(x, y) + \tau_{[x, y]} = [\tau_x, \tau_y] + \psi_2(x, y)$$

We now define vertical and horizontal composition for these 2-homomorphisms. First let  $V$  and  $V'$  be 2-term  $L_\infty$ -algebras and let  $\phi, \psi, \gamma: V \rightarrow V'$  be  $L_\infty$ -homomorphisms. If  $\theta: \phi \Rightarrow \psi$  and  $\tau: \psi \Rightarrow \gamma$  are  $L_\infty$ -2-homomorphisms, we define their **vertical** composite,  $\theta\tau: \phi \rightarrow \gamma$ , by

$$\theta\tau(x) := \theta(x) + \tau(x).$$

Next, let  $V, V', V''$  be 2-term  $L_\infty$ -algebras and let  $\phi, \psi: V \rightarrow V'$  and  $\phi', \psi': V' \rightarrow V''$  be  $L_\infty$ -homomorphisms. If  $\tau: \phi \Rightarrow \psi$  and  $\tau': \phi' \Rightarrow \psi'$  are  $L_\infty$ -2-homomorphisms, we define their **horizontal composite**,  $\tau \circ \tau': \phi\phi' \Rightarrow \psi\psi'$ , in either of two equivalent ways:

$$\begin{aligned}\tau \circ \tau'(x) &:= \tau'(\phi_0(x)) + \psi'_1(\tau(x)) \\ &= \phi'_1(\tau(x)) + \tau'(\psi_0(x)).\end{aligned}$$

Finally, given a  $L_\infty$ -homomorphism  $\phi: V \rightarrow V'$ , the identity  $L_\infty$ -2-homomorphism  $1_\phi: \phi \Rightarrow \phi$  is given by  $1_\phi(x) = 1_{\phi_0(x)}$ .

With these definitions, it becomes a straightforward exercise to prove the following:

**Proposition 33.** *There is a strict 2-category  $\mathbf{2Term}L_\infty$  with 2-term  $L_\infty$ -algebras as objects, homomorphisms between these as morphisms, and 2-homomorphisms between those as 2-morphisms.*

With these definitions one can strengthen Theorem 31 as follows:

**Theorem 34.** *The 2-categories  $\mathbf{Lie2Alg}$  and  $\mathbf{2Term}L_\infty$  are 2-equivalent.*

The main benefit of the results in this section is that they provide us with another method to create examples of Lie 2-algebras. Instead of thinking of a Lie 2-algebra as a category equipped with extra structure, we may work with a 2-term chain complex endowed with the structure described in Lemma 28. In the next two sections we investigate the results of trivializing various aspects of a Lie 2-algebra, or equivalently of the corresponding 2-term  $L_\infty$ -algebra.

## 2.4 Strict Lie 2-algebras

As the reader may expect, a ‘strict’ Lie 2-algebra is a mixture of a Lie algebra and category in which all laws hold on the nose as equations, not just up to isomorphism. In one of his papers [B], Baez showed how to construct these starting from ‘strict Lie 2-groups’. Here we describe this process in a somewhat more highbrow manner, and explain how these ‘strict’ notions are special cases of the semistrict ones described here.

Since we only weakened the Jacobi identity in our definition of ‘semistrict’ Lie 2-algebra, we need only require that the Jacobiator be the identity to recover the ‘strict’ notion:

**Definition 35.** *A semistrict Lie 2-algebra is **strict** if its Jacobiator is the identity.*

Similarly, requiring that the bracket be strictly preserved gives the notion of ‘strict’ homomorphism between Lie 2-algebras:

**Definition 36.** *Given semistrict Lie 2-algebras  $L$  and  $L'$ , a homomorphism  $F: L \rightarrow L'$  is **strict** if  $F_2$  is the identity.*

**Proposition 37.** *Lie2Alg contains a sub-2-category **SLie2Alg** with strict Lie 2-algebras as objects, strict homomorphisms between these as morphisms, and arbitrary 2-homomorphisms between those as 2-morphisms.*

**Proof.** One need only check that if the homomorphisms  $F: L \rightarrow L'$  and  $G: L' \rightarrow L''$  have  $F_2 = 1, G_2 = 1$ , then their composite has  $(FG)_2 = 1$ .  $\square$

The following proposition shows that strict Lie 2-algebras as defined here agree with those as defined previously [B]:

**Proposition 38.** *The 2-category SLie2Alg is isomorphic to the 2-category LieAlgCat consisting of categories, functors and natural transformations in LieAlg.*

**Proof.** This is just a matter of unravelling the definitions.  $\square$

Just as Lie groups have Lie algebras, ‘strict Lie 2-groups’ have strict Lie 2-algebras. Before we can state this result precisely, we must recall the concept of a strict Lie 2-group, which was treated in greater detail in HDA5 [BLau]:

**Definition 39.** *We define **SLie2Grp** to be the strict 2-category LieGrpCat consisting of categories, functors and natural transformations in LieGrp. We call the objects in this 2-category **strict Lie 2-groups**; we call the morphisms between these **strict homomorphisms**, and we call the 2-morphisms between those **2-homomorphisms**.*

**Proposition 40.** *There exists a unique 2-functor*

$$d: \text{SLie2Grp} \rightarrow \text{SLie2Alg}$$

*such that:*

1.  *$d$  maps any strict Lie 2-group  $C$  to the strict Lie 2-algebra  $dC = \mathfrak{c}$  for which  $\mathfrak{c}_0$  is the Lie algebra of the Lie group of objects  $C_0$ ,  $\mathfrak{c}_1$  is the Lie algebra of the Lie group of morphisms  $C_1$ , and the maps  $s, t: \mathfrak{c}_1 \rightarrow \mathfrak{c}_0$ ,  $i: \mathfrak{c}_0 \rightarrow \mathfrak{c}_1$  and  $\circ: \mathfrak{c}_1 \times_{\mathfrak{c}_0} \mathfrak{c}_1 \rightarrow \mathfrak{c}_1$  are the differentials of those for  $C$ .*
2.  *$d$  maps any strict Lie 2-group homomorphism  $F: C \rightarrow C'$  to the strict Lie 2-algebra homomorphism  $dF: \mathfrak{c} \rightarrow \mathfrak{c}'$  for which  $(dF)_0$  is the differential of  $F_0$  and  $(dF)_1$  is the differential of  $F_1$ .*
3.  *$d$  maps any strict Lie 2-group 2-homomorphism  $\alpha: F \Rightarrow G$  where  $F, G: C \rightarrow C'$  to the strict Lie 2-algebra 2-homomorphism  $d\alpha: dF \Rightarrow dG$  for which the map  $d\alpha: \mathfrak{c}_0 \rightarrow \mathfrak{c}_1$  is the differential of  $\alpha: C_0 \rightarrow C_1$ .*

**Proof.** The proof of this long-winded proposition is a quick exercise in internal category theory: the well-known functor from LieGrp to LieAlg preserves pullbacks, so it maps categories, functors and natural transformations in LieGrp to those in LieAlg, defining a 2-functor  $d: \text{SLie2Grp} \rightarrow \text{SLie2Alg}$ , which is given explicitly as above.  $\square$

We would like to generalize this theorem and define the Lie 2-algebra not just of a strict Lie 2-group, but of a general Lie 2-group as defined in HDA5 [BLau]. However, this may require a weaker concept of Lie 2-algebra than that studied here.

## 2.5 Skeletal Lie 2-algebras

A semistrict Lie 2-algebra is *strict* when the Jacobiator is the identity, which means that the map  $l_3$  vanishes in the corresponding  $L_\infty$ -algebra. We now investigate the consequences of assuming the differential  $d$  vanishes in the corresponding  $L_\infty$ -algebra. Thanks to the formula

$$d\vec{f} = t(f) - s(f),$$

this implies that the source of any morphism in the Lie 2-algebra equals its target. In other words, the Lie 2-algebra is *skeletal*:

**Definition 41.** *A category is **skeletal** if isomorphic objects are equal.*

Skeletal categories are useful in category theory because every category is equivalent to a skeletal one formed by choosing one representative of each isomorphism class of objects [M]. The same sort of thing is true in the context of 2-vector spaces:

**Lemma 42.** *Any 2-vector space is equivalent, as an object of  $2\mathbf{Vect}$ , to a skeletal one.*

**Proof.** Using the result of Theorem 12 we may treat our 2-vector spaces as 2-term chain complexes. In particular, a 2-vector space is skeletal if the corresponding 2-term chain complex has vanishing differential, and two 2-vector spaces are equivalent if the corresponding 2-term chain complexes are chain homotopy equivalent. So, it suffices to show that any 2-term chain complex is chain homotopy equivalent to one with vanishing differential. This is well-known, but the basic idea is as follows. Given a 2-term chain complex

$$C_1 \xrightarrow{d} C_0$$

we express the vector spaces  $C_0$  and  $C_1$  as  $C_0 = \text{im}(d) \oplus C'_0$  and  $C_1 = \ker(d) \oplus X$  where  $X$  is a vector space complement to  $\ker(d)$  in  $C_1$ . This allows us to define a 2-term chain complex  $C'$  with vanishing differential:

$$C'_1 = \ker(d) \xrightarrow{0} C'_0.$$

The inclusion of  $C'$  in  $C$  can easily be extended to a chain homotopy equivalence.

□

Using this fact we obtain a result that will ultimately allow us to classify Lie 2-algebras:

**Proposition 43.** *Every Lie 2-algebra is equivalent, as an object of  $\mathbf{Lie2Alg}$ , to a skeletal one.*

**Proof.** Given a Lie 2-algebra  $L$  we may use Lemma 42 to find an equivalence between the underlying 2-vector space of  $L$  and a skeletal 2-vector space  $L'$ . We may then use this to transport the Lie 2-algebra structure from  $L$  to  $L'$ , and obtain an equivalence of Lie 2-algebras between  $L$  and  $L'$ . □

It is interesting to observe that a skeletal Lie 2-algebra that is also strict amounts to nothing but a Lie algebra  $L_0$  together with a representation of  $L_0$  on a vector space  $L_1$ . This is the infinitesimal analogue of how a strict skeletal 2-group  $G$  consists of a group  $G_0$  together with an action of  $G_0$  as automorphisms of an abelian group  $G_1$ . Thus, the representation theory of groups and Lie algebras is automatically subsumed in the theory of 2-groups and Lie 2-algebras!

To generalize this observation to other skeletal Lie 2-algebras, we recall some basic definitions concerning Lie algebra cohomology:

**Definition 44.** Let  $\mathfrak{g}$  be a Lie algebra and  $\rho$  a representation of  $\mathfrak{g}$  on the vector space  $V$ . Then a  **$V$ -valued  $n$ -cochain**  $\omega$  on  $\mathfrak{g}$  is a totally antisymmetric map

$$\omega: \mathfrak{g}^{\otimes n} \rightarrow V.$$

The vector space of all  $n$ -cochains is denoted by  $C^n(\mathfrak{g}, V)$ . The **coboundary operator**  $\delta: C^n(\mathfrak{g}, V) \rightarrow C^{n+1}(\mathfrak{g}, V)$  is defined by:

$$\begin{aligned} (\delta\omega)(v_1, v_2, \dots, v_{n+1}) &:= \sum_{i=1}^{n+1} (-1)^{i+1} \rho(v_i) \omega(v_1, \dots, \hat{v}_i, \dots, v_{n+1}) \\ &+ \sum_{1 \leq j < k \leq n+1} (-1)^{j+k} \omega_n([v_j, v_k], v_1, \dots, \hat{v}_j, \dots, \hat{v}_k, \dots, v_{n+1}), \end{aligned}$$

**Proposition 45.** The Lie algebra coboundary operator  $\delta$  satisfies  $\delta^2 = 0$ .

**Definition 46.** A  $V$ -valued  $n$ -cochain  $\omega$  on  $\mathfrak{g}$  is an  **$n$ -cocycle** when  $\delta\omega = 0$  and an  **$n$ -coboundary** if there exists an  $(n-1)$ -cochain  $\theta$  such that  $\omega = \delta\theta$ . We denote the groups of  $n$ -cocycles and  $n$ -coboundaries by  $Z^n(\mathfrak{g}, V)$  and  $B^n(\mathfrak{g}, V)$  respectively. The  $n$ th **Lie algebra cohomology group**  $H^n(\mathfrak{g}, V)$  is defined by

$$H^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V) / B^n(\mathfrak{g}, V).$$

The following result illuminates the relationship between Lie algebra cohomology and  $L_\infty$ -algebras.

**Theorem 47.** There is a one-to-one correspondence between  $L_\infty$ -algebras consisting of only two nonzero terms  $V_0$  and  $V_n$ , with  $d = 0$ , and quadruples  $(\mathfrak{g}, V, \rho, l_{n+2})$  where  $\mathfrak{g}$  is a Lie algebra,  $V$  is a vector space,  $\rho$  is a representation of  $\mathfrak{g}$  on  $V$ , and  $l_{n+2}$  is a  $(n+2)$ -cocycle on  $\mathfrak{g}$  with values in  $V$ .

**Proof.**

Given such an  $L_\infty$ -algebra  $V$  we set  $\mathfrak{g} = V_0$ .  $V_0$  comes equipped with a bracket as part of the  $L_\infty$ -structure, and since  $d$  is trivial, this bracket satisfies the Jacobi identity on the nose, making  $\mathfrak{g}$  into a Lie algebra. We define  $V = V_n$ , and note that the bracket also gives a map  $\rho: \mathfrak{g} \otimes V \rightarrow V$ , defined by  $\rho(x)f = [x, f]$  for  $x \in \mathfrak{g}, f \in V$ . We have

$$\rho([x, y])f = [[x, y], f]$$



$$\begin{aligned}
&= [[x, f], y] + [x, [y, f]] \quad \text{by (2) of Definition 27} \\
&= [\rho(x)f, y] + [x, \rho(y)f]
\end{aligned}$$

for all  $x, y \in \mathfrak{g}$  and  $f \in V$ , so that  $\rho$  is a representation. Finally, the  $L_\infty$  structure gives a map  $l_{n+2}: \mathfrak{g}^{\otimes(n+2)} \rightarrow V$  which is in fact a  $(n+2)$ -cocycle. To see this, note that

$$0 = \sum_{i+j=n+4} \sum_{\sigma} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n+2)})$$

where we sum over  $(i, (n+3)-i)$ -unshuffles  $\sigma \in S_{n+3}$ . However, the only choices for  $i$  and  $j$  that lead to nonzero  $l_i$  and  $l_j$  are  $i = n+2, j = 2$  and  $i = 2, j = n+2$ . In addition, notice that in this situation,  $\chi(\sigma)$  will consist solely of the sign of the permutation because all of our  $x_i$ 's have degree zero. Thus, the above becomes, with  $\sigma$  a  $(n+2, 1)$ -unshuffle and  $\tau$  a  $(2, n+1)$ -unshuffle:

$$\begin{aligned}
0 &= \sum_{\sigma} \chi(\sigma) (-1)^{n+2} [l_{n+2}(x_{\sigma(1)}, \dots, x_{\sigma(n+2)}), x_{\sigma(n+3)}] \\
&\quad + \sum_{\tau} \chi(\tau) l_{n+2}([x_{\tau(1)}, x_{\tau(2)}], x_{\tau(3)}, \dots, x_{\tau(n+3)}) \\
&= \sum_{i=1}^{n+3} (-1)^{n+3-i} (-1)^{n+2} [l_{n+2}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+3}), x_i] \\
&\quad + \sum_{1 \leq i < j \leq n+3} (-1)^{i+j+1} l_{n+2}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+3}) \quad (\dagger) \\
&= \sum_{i=1}^{n+3} (-1)^{i+1} [l_{n+2}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+3}), x_i] \\
&\quad + \sum_{1 \leq i < j \leq n+3} (-1)^{i+j+1} l_{n+2}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+3}) \\
&= - \sum_{i=1}^{n+3} (-1)^{i+1} [x_i, l_{n+2}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+3})] \\
&\quad - \sum_{1 \leq i < j \leq n+3} (-1)^{i+j} l_{n+2}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+3}) \\
&= -\delta l_{n+2}(x_1, x_2, \dots, x_{n+3}).
\end{aligned}$$

The first sum in  $(\dagger)$  follows because we have  $(n+3)$   $(n+2, 1)$ -unshuffles and the sign of any such unshuffle is  $(-1)^{n+3-i}$ . The second sum follows similarly because we have  $(n+3)$   $(2, n+1)$ -unshuffles and the sign of a  $(2, n+1)$ -unshuffle is  $(-1)^{i+j+1}$ . Therefore,  $l_{n+2}$  is a  $(n+2)$ -cocycle.

Conversely, given a Lie algebra  $\mathfrak{g}$ , a representation  $\rho$  of  $\mathfrak{g}$  on a vector space  $V$ , and an  $(n+2)$ -cocycle  $l_{n+2}$  on  $\mathfrak{g}$  with values in  $V$ , we define our  $L_\infty$ -algebra  $V$  by setting  $V_0 = \mathfrak{g}$ ,  $V_n = V$ ,  $V_i = \{0\}$  for  $i \neq 0, n$  and  $d = 0$ . It remains to define the system of linear maps

$l_k$ , which we do as follows: Since  $\mathfrak{g}$  is a Lie algebra, we have a bracket defined on  $V_0$ . We extend this bracket to define the map  $l_2$ , denoted by  $[\cdot, \cdot]: V_i \otimes V_j \rightarrow V_{i+j}$  where  $i, j = 0, n$ , as follows:

$$\begin{aligned} [x, f] &= \rho(x)f, \\ [f, y] &= -\rho(y)f, \\ [f, g] &= 0 \end{aligned}$$

for  $x, y \in V_0$  and  $f, g \in V_n$ . With this definition, the map  $[\cdot, \cdot]$  satisfies condition (1) of Definition 27. We define  $l_k = 0$  for  $3 \leq k \leq n+1$  and  $k > n+2$ , and take  $l_{n+2}$  to be the given  $(n+2)$  cocycle, which satisfies conditions (1) and (2) of Definition 27 by the cocycle condition.  $\square$

We can classify skeletal Lie 2-algebras using the above construction with  $n = 1$ :

**Corollary 48.** *There is a one-to-one correspondence between isomorphism classes of skeletal Lie 2-algebras and isomorphism classes of quadruples consisting of a Lie algebra  $\mathfrak{g}$ , a vector space  $V$ , a representation  $\rho$  of  $\mathfrak{g}$  on  $V$ , and a 3-cocycle on  $\mathfrak{g}$  with values in  $V$ .*

**Proof.** This is immediate from Theorem 31 and Theorem 47.  $\square$

Since every Lie 2-algebra is equivalent as an object of  $\text{Lie2Alg}$  to a skeletal one, this in turn lets us classify *all* Lie 2-algebras, though only up to equivalence:

**Theorem 49.** *There is a one-to-one correspondence between equivalence classes of Lie 2-algebras (where equivalence is as objects of the 2-category  $\text{Lie2Alg}$ ) and isomorphism classes of quadruples consisting of a Lie algebra  $\mathfrak{g}$ , a vector space  $V$ , a representation  $\rho$  of  $\mathfrak{g}$  on  $V$ , and an element of  $H^3(\mathfrak{g}, V)$ .*

**Proof.** This follows from Theorem 43 and Corollary 48; we leave it to the reader to verify that equivalent skeletal Lie 2-algebras give cohomologous 3-cocycles and conversely.  $\square$

We conclude with perhaps the most interesting examples of finite-dimensional Lie 2-algebras coming from Theorem 48. These make use of the following identities involving the Killing form  $\langle x, y \rangle := \text{tr}(\text{ad}(x)\text{ad}(y))$  of a finite-dimensional Lie algebra:

$$\langle x, y \rangle = \langle y, x \rangle,$$

and

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle.$$

**Example 50.** *There is a skeletal Lie 2-algebra built using Theorem 48 by taking  $V_0 = \mathfrak{g}$  to be a finite-dimensional Lie algebra over the field  $k$ ,  $V_1$  to be  $k$ ,  $\rho$  the trivial representation, and  $l_3(x, y, z) = \langle x, [y, z] \rangle$ . We see that  $l_3$  is a 3-cocycle using the above identities as follows:*

$$\begin{aligned} (\delta l_3)(w, x, y, z) &= \rho(w)l_3(x, y, z) - \rho(x)l_3(w, y, z) + \rho(y)l_3(w, x, z) - \rho(z)l_3(w, x, y) \\ &\quad - l_3([w, x], y, z) + l_3([w, y], x, z) - l_3([w, z], x, y) \\ &\quad - l_3([x, y], w, z) + l_3([x, z], w, y) - l_3([y, z], w, x) \end{aligned}$$

$$\begin{aligned}
&= -\langle [w, x], [y, z] \rangle + \langle [w, y], [x, z] \rangle - \langle [w, z], [x, y] \rangle \\
&\quad - \langle [x, y], [w, z] \rangle + \langle [x, z], [w, y] \rangle - \langle [y, z], [w, x] \rangle
\end{aligned}$$

This second step above follows because we have a trivial representation. Continuing on, we have

$$\begin{aligned}
(\delta l_3)(w, x, y, z) &= -2\langle [w, x], [y, z] \rangle + 2\langle [w, y], [x, z] \rangle - 2\langle [w, z], [x, y] \rangle \\
&= -2\langle w, [x, [y, z]] \rangle + 2\langle w, [y, [x, z]] \rangle - 2\langle w, [z, [x, y]] \rangle \\
&= -2\langle w, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \rangle \\
&= -2\langle w, 0 \rangle \\
&= 0.
\end{aligned}$$

More generally, we obtain a Lie 2-algebra this way taking  $l_3(x, y, z) = \hbar \langle x, [y, z] \rangle$  where  $\hbar$  is any element of  $k$ . We call this Lie 2-algebra  $\mathfrak{g}_\hbar$ .

It is well known that the Killing form of  $\mathfrak{g}$  is nondegenerate if and only if  $\mathfrak{g}$  is semisimple. In this case the 3-cocycle described above represents a nontrivial cohomology class when  $\hbar \neq 0$ , so by Theorem 49 the Lie 2-algebra  $\mathfrak{g}_\hbar$  is not equivalent to a skeletal one with vanishing Jacobiator. In other words, we obtain a Lie 2-algebra that is not equivalent to a skeletal strict one.

Suppose the field  $k$  has characteristic zero, the Lie algebra  $\mathfrak{g}$  is finite dimensional and semisimple, and  $V$  is finite dimensional. Then a version of Whitehead's Lemma [AIP] says that  $H^3(\mathfrak{g}, V) = \{0\}$  whenever the representation of  $\mathfrak{g}$  on  $V$  is nontrivial and irreducible. This places some limitations on finding interesting examples of nonstrict Lie 2-algebras other than those of the form  $\mathfrak{g}_\hbar$ .

We expect the Lie 2-algebras  $\mathfrak{g}_\hbar$  to be related to quantum groups, affine Lie algebras and other constructions that rely crucially on the 3-cocycle  $\langle x, [y, z] \rangle$  or the closely related 2-cocycle on  $\mathfrak{g}[z, z^{-1}]$ . The smallest example comes from  $\mathfrak{su}(2)$ . Since  $\mathfrak{su}(2)$  is isomorphic to  $\mathbb{R}^3$  with its usual vector cross product, and its Killing form is proportional to the dot product, this Lie 2-algebra relies solely on familiar properties of the dot product and cross product:

$$\begin{aligned}
x \times y &= -y \times x, \\
x \cdot y &= y \cdot x, \\
x \cdot (y \times z) &= (x \times y) \cdot z, \\
x \times (y \times z) + y \times (z \times x) + z \times (x \times y) &= 0.
\end{aligned}$$

It will be interesting to see if this Lie 2-algebra, where the Jacobiator comes from the triple product, has any applications to physics. Just for fun, we work out the details again in this case:

**Example 51.** *There is a skeletal Lie 2-algebra built using Theorem 48 by taking  $V_0 = \mathbb{R}^3$  equipped with the cross product,  $V_1 = \mathbb{R}$ ,  $\rho$  the trivial representation, and  $l_3(x, y, z) = x \cdot (y \times z)$ . We see that  $l_3$  is a 3-cocycle as follows:*

$$(\delta l_3)(w, x, y, z) = -l_3([w, x], y, z) + l_3([w, y], x, z) - l_3([w, z], x, y)$$

$$\begin{aligned}
& -l_3([x, y], w, z) + l_3([x, z], w, y) - l_3([y, z], w, x) \\
= & -(w \times x) \cdot (y \times z) + (w \times y) \cdot (x \times z) - (w \times z) \cdot (x \times y) \\
& -(x \times y) \cdot (w \times z) + (x \times z) \cdot (w \times y) - (y \times z) \cdot (w \times x) \\
= & -2(w \times x) \cdot (y \times z) + 2(w \times y) \cdot (x \times z) - 2(w \times z) \cdot (x \times y) \\
= & -2w \cdot (x \times (y \times z)) + 2w \cdot (y \times (x \times z)) - 2w \cdot (z \times (x \times y)) \\
= & -2w \cdot (x \times (y \times z) + y \times (z \times x) + z \times (x \times y)) \\
= & 0.
\end{aligned}$$

We now shift gears by moving away from the categorification process and considering the passage from Lie groups to Lie algebras. In the next chapter, we describe a new method of obtaining the Lie algebra of a Lie group using algebraic structures called ‘quandles’.

## Chapter 3

# Lie Theory, Quandles and Braids

Now that we have a theory of Lie 2-algebras as well as that of Lie 2-groups, it becomes natural to wonder if the standard results of Lie theory have higher-dimensional analogues. The obvious first question is whether Lie 2-algebras arise from Lie 2-groups. Of course, as mentioned in the previous chapter, this may require a weaker version of our semistrict Lie 2-algebras. It is our hope that the correct definition of a weak Lie 2-algebra will present itself as we unravel the passage of Lie 2-group to Lie 2-algebra. Once we have established that Lie 2-groups give rise to Lie 2-algebras, we would then like to categorify the exponential map, and determine whether or not a homomorphism between Lie 2-groups induces a homomorphism between the corresponding weak Lie 2-algebras.

In this chapter we begin to explore the first question above. In this direction, we describe a novel method of obtaining Lie algebras from Lie groups, which we intend to categorify to show that every Lie 2-algebra arises from a Lie 2-group. The standard procedure for obtaining a Lie algebra from a Lie group consists of taking the tangent space of the Lie group at the identity element. Moreover, one often then shows that the space of left-invariant vector fields on the Lie group is isomorphic as a vector space to this tangent space at the identity. As we do not, yet, have notions of categorified vector fields, we seek an alternate route.

Since the Lie algebra is a tangent space of the Lie group, it is natural to obtain operations on the Lie algebra from those on the Lie group by differentiation. Considering that negatives in the Lie algebra correspond to inverses in the Lie group, one may be tempted to think that the bracket in the Lie algebra arises from differentiating the multiplication. However, it turns out that the differential of the product map is simply the addition operation in the Lie algebra. The bracket results from differentiating conjugation *twice*, which can be seen in the following computation: Let  $G$  be a Lie group and  $\mathfrak{g}$  its Lie algebra. Consider curves  $\exp(sx)$  and  $\exp(ty)$  in  $G$  where  $x, y \in \mathfrak{g}$ . Then, if  $G$  is a matrix Lie group, we have:

$$[x, y] = \frac{d^2}{dsdt} (e^{sx} e^{ty} e^{-sx})|_{s=t=0}$$

Therefore, it is the operation of *conjugation*, and not multiplication, in the Lie group which gives the bracket operation. Our new description of the passage from a Lie group to its Lie algebra captures this key aspect.

The first step in this process involves focusing our attention on the conjugation operation. Consider two elements  $g$  and  $h$  in some group  $G$ . If we denote left conjugation by  $g$ ,  $ghg^{-1}$ , as  $g \triangleright h$  and right conjugation by  $g$ ,  $g^{-1}hg$ , as  $h \triangleleft g$  then Joyce has shown that all equational laws involving only left and right conjugation can be derived from these three [J]:

- (idempotence)  $g \triangleright g = g$
- (inverses)  $(g \triangleright h) \triangleleft g = h = g \triangleright (h \triangleleft g)$
- (self-distributivity)  $g \triangleright (h \triangleright k) = (g \triangleright h) \triangleright (g \triangleright k)$

for  $g, h, k$  in some group  $G$ . That is, conjugating an element by itself acts as the identity, left and right conjugation are inverses, and conjugation satisfies a self-distributive law. These observations led to the creation of a new algebraic structure: a ‘quandle’. A quandle is a set  $Q$  together with two binary operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$  satisfying the three identities above. As anticipated, the primordial example of a quandle is a group with the two operations of left and right conjugation.

Since every law involving conjugation can be derived from the three axioms above, the theory of quandles can be regarded as the theory of conjugation. Thus, since conjugation in a Lie group gives rise to the bracket operation in its Lie algebra, we desire a way to treat our Lie groups as though they were quandles. In fact, we can get away with less than a quandle; a structure we call a ‘spindle’ will do.

We begin this chapter in Section 3.1.1 by recalling the definitions of quandles and racks and introducing two related notions: ‘spindles’ and ‘shelves’. We demonstrate the relationship to groups and define the categories of Shelf, Rack, Spind, and Quand.

In Section 3.1.2, we illustrate the connection between these four algebraic concepts and topology. In particular, we demonstrate a relationship to the Reidemeister moves, and show that both quandles and Lie algebras give solutions to the Yang–Baxter equation. This suggests that a Lie algebra is perhaps a quandle! We investigate this suggestion later on. Furthermore, we show that in some sense, the inverse properties are more essential than the idempotence and self-distributivity axioms. This is because the latter two have completely symmetrical counterparts that are implied by the three axioms given above, whereas *neither* inverse property implies the other.

We continue in Section 3.1.3 by describing the connection between these four algebraic concepts and various braid and framed braid monoids and groups. It turns out that quandles, racks, spindles, and shelves, respectively, give an action of the braid group, framed braid group, positive braid monoid, and positive framed braid monoid! This is a consequence of the fact that each of the three quandle axioms is equivalent, in a suitable context, to one of the three Reidemeister moves.

Clearly, a Lie group may be thought of as quandle in the same way that an ordinary group can, via conjugation. However, this description lacks knowledge about the manifold structure of our Lie group. This deficiency suggests that we require a means by which we can describe Lie groups in terms of special sorts of quandles — ones that are also manifolds! Thus, in Section 3.1.4 we internalize these four concepts, which will allow us to treat Lie groups as ‘quandles in  $\text{Diff}_*$ ’, the category of pointed smooth manifolds. Just as a group

gives a quandle, we will use internalization to show that a group in any category  $K$  with products is a special sort of ‘quandle in  $K$ ’. Using the language of internalization, Lie groups, which are groups in  $\text{Diff}_*$ , will become quandles in  $\text{Diff}_*$ .

Now that we have a way to think of our Lie groups as quandles, which allows us to focus on conjugation, we must bring differentiation into the game in order to obtain Lie algebras. Since the bracket arises from differentiating conjugation twice, it will not suffice to simply consider the tangent space of our Lie group at the identity. So instead we consider the symmetric algebra on  $T_*G$ . This has two desirable qualities, namely that the process of taking the symmetric algebra of a space preserves products, and the symmetric algebra contains information about the space  $S^2(T_*G)$ , which can be thought of as the space of second-order Taylor expansions of polynomials in the elements of  $T_*G$ . However, like the tangent space, the symmetric algebra has its flaws as well. The problem is that a map  $f: M \rightarrow N$  gives a map from  $S(T_pM)$  to  $S(T_{f(p)}N)$  that only depends on  $df$  and *not* on the higher derivatives of  $f$ , which is what we need in order to obtain the bracket.

Therefore, in Section 3.2.1, we recall the concept of ‘ $k$ -jets’. Roughly speaking, a  $k$ -jet of a smooth function  $f$  is a gadget which keeps track of the value of the function at a point together with all derivatives of  $f$  up to the  $k$ th order at that point. That is, this is simply a way of describing the Taylor expansion of  $f$  at a given point up to order  $k$ . The algebra,  $J^k(M, p)$ , of  $k$ -jets of real-valued functions on  $M$  at the point  $p$ , has the desirable properties of keeping track of the necessary derivatives, and being isomorphic to

an algebra that we are familiar with:  $\bigoplus_{i=0}^k S^i(T_p^*M)$ . Nonetheless, this space of  $k$ -jets is still

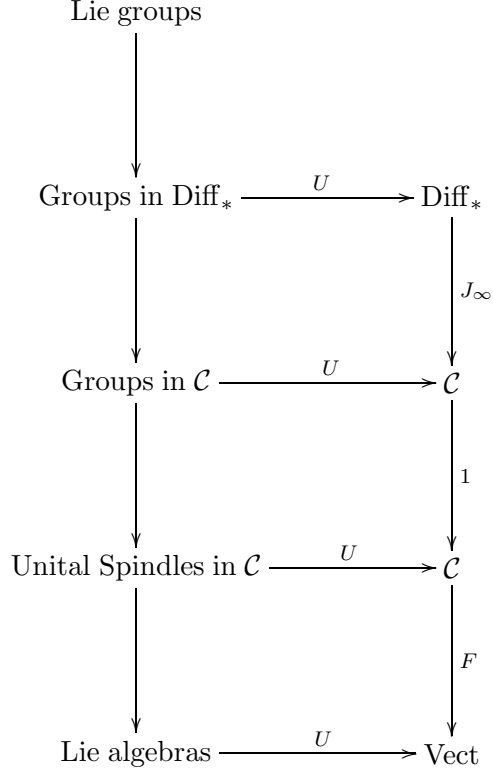
not quite what we need, since it defines a *contravariant* functor from  $\text{Diff}_*$ , the category of smooth manifolds, to  $\text{Vect}$ , the category of vector spaces, whereas we want a *covariant* functor. Therefore, we recall the notion of the coalgebra of ‘ $k$ -cojets’,  $J_k(M, p)$ , which is defined to be the dual of  $J^k(M, p)$ . While  $J^k: \text{Diff}_* \rightarrow \text{Vect}$  does define a covariant functor, it does not preserve products, so  $J^k$  will not send groups and quandles in  $\text{Diff}_*$  to groups and quandles in  $\text{Vect}$ , which we require.

Thus, we continue in this section by recalling the concept of the space of ‘cojets’,  $J_\infty(M, p)$ , which is simply the union over  $k$  of the spaces of  $k$ -cojets.  $J_\infty(M, p)$  forms a cocommutative coalgebra that is isomorphic as a coalgebra to  $S(T_pM)$ , which allows us in Section 3.2.2 to define a covariant functor  $J_\infty: \text{Diff}_* \rightarrow \mathcal{C}$  where  $\mathcal{C}$  is the category of cocommutative coalgebras that are isomorphic as coalgebras to  $S(V)$  for some vector space  $V$ . That is, the objects of this category are triples  $(C, V, \alpha)$ , where  $C$  is a coalgebra,  $V$  is a vector space, and  $\alpha$  is a coalgebra isomorphism from  $C$  to  $S(V)$ . We call the objects of  $\mathcal{C}$  ‘special coalgebras’. We show that  $\mathcal{C}$  has products, which enables us to define a group in  $\mathcal{C}$ , and then demonstrate that  $J_\infty$  preserves products which implies that it sends groups in  $\text{Diff}_*$  to groups in  $\mathcal{C}$ .

Finally, in Section 3.2.4 we use our special coalgebra  $(C, V, \alpha)$  to obtain the Lie algebra of the Lie group we started with. This requires defining a functor  $F: \mathcal{C} \rightarrow \text{Vect}$  that ‘picks off’ the vector space. Then, given a Lie group  $G$ , we use the fact that  $(J_\infty(G, 1), T_1G, \alpha)$  is a group in  $\mathcal{C}$ , and hence quandle in  $\mathcal{C}$  to define the bracket on  $T_1G$  in terms of the quandle operation  $\triangleright$  on  $J_\infty(G, 1)$ . The Jacobi identity for the bracket follows from the self-distributive law for the quandle operation, while the antisymmetry of

the bracket comes from the idempotence law satisfied by the quandle operation. We again remark that we do not require a full-fledged quandle for this process, but can get by with a ‘spindle’ since we use only the self-distributivity and idempotence conditions.

We summarize our process in the following diagram:



In the final section of this chapter, we conclude by describing the categorified version of this process and present an outline of our future work.

We remind the reader that in all that follows, we denote the composition of two morphisms  $f: x \rightarrow y$  and  $g: y \rightarrow z$  as  $fg: x \rightarrow z$ . In addition,  $\pi_1$  and  $\pi_2$  are the canonical projections of  $Q \times Q$  onto the first and second components, respectively.

## 3.1 Shelves, Racks, Spindles and Quandles

### 3.1.1 Definitions and Relation to Groups

Quandles and racks are algebraic structures that capture the essential properties of the operations of conjugation in a group. While both provide interesting invariants of braids, quandles also give a conceptual explanation of the passage from a Lie group to its Lie algebra. This is a result of the fact that the bracket in the Lie algebra arises from differentiating conjugation in the Lie group. Our goal is to explain this passage in a way which might make it easier to categorify.

A comprehensive history of racks, beginning with the unpublished correspondence between Conway and Wraith roughly 45 years ago, is provided in the work of Fenn and



Rourke [FR]. Quandles, which are special cases of racks, were first introduced as a source of knot invariants by David Joyce in 1982 and have been studied and explored since then in various papers [Bri, FR, J, K]. Fenn and Rourke demonstrate that racks provide an elegant algebraic framework which gives complete algebraic invariants for both framed links in 3-manifolds and for the 3-manifolds themselves. Likewise, Joyce uses the structure provided by quandles to formulate additional invariants. In fact, Fenn and Rourke suggest that the theory of racks presents the opportunity to find a complete sequence of computable invariants for framed links and 3-manifolds.

Both quandles and racks illustrate the connection between algebra and topology since their axioms algebraically encode the three Reidemeister moves. In particular, we will show in the next section that the ‘self-distributivity axiom’ is equivalent to the Yang–Baxter equation, or third Reidemeister move, in a suitable context.

We begin with the simplest of these four structures, a ‘shelf’:

**Definition 52.** *A left shelf  $(S, \triangleright)$  is a nonempty set  $S$  equipped with a binary operation  $\triangleright: S \times S \rightarrow S$  called **left conjugation**, which satisfies*

*(i) (left distributive law)  $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$  for all  $x, y, z \in S$ .*

A typical example of a left shelf is a group  $G$  where the shelf operation is defined to be left conjugation,  $x \triangleright y := xyx^{-1}$  for all  $x, y \in G$ . A trivial computation shows that the left distributive law holds for this operation and that it amounts to the distributive law satisfied by conjugation. As we have alluded to in the introduction to this chapter, it is precisely this self-distributive law that corresponds to the Jacobi identity in a Lie algebra once we differentiate it.

Since a group can also act on itself by right conjugation,  $x^{-1}yx$ , which satisfies an analogous distributive law, we obtain the symmetrical notion of a ‘right shelf’:

**Definition 53.** *A right shelf  $(S, \triangleleft)$  is a nonempty set  $S$  equipped with a binary operation  $\triangleleft: S \times S \rightarrow S$  called **right conjugation**, which satisfies*

*(ii) (right distributive law)  $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$  for all  $x, y, z \in S$ .*

As mentioned above, an example of a right shelf is a group  $G$  where the shelf operation is defined to be right conjugation,  $y \triangleleft x := x^{-1}yx$  for all  $x, y \in G$ .

Henceforth, the term **shelf** will always refer to a left shelf. We remark that, in general, the shelf operation is neither associative nor commutative. Furthermore, we should think of these shelf operations as *actions*, where the element pointed to is the one being acted upon. The location of the remaining element determines whether we have left or right conjugation. For instance, ‘ $x \triangleleft y$ ’ says that  $x$  is being conjugated on the right by  $y$ .

In the next section, we prove that we can define a Yang–Baxter operator on a shelf, thus demonstrating that the left distributive law is equivalent to the Yang–Baxter equation, or third Reidemeister move. We now turn to the concept of a ‘rack’, which is a set equipped with two operations satisfying properties corresponding to the second and third Reidemeister moves. Roughly speaking, a rack results when a left shelf and a right shelf fit together nicely.

**Definition 54.** A rack  $(Q, \triangleright, \triangleleft)$  is a nonempty set  $Q$  together with two binary operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$ , called **left conjugation** and **right conjugation**, such that  $(Q, \triangleright)$  is a left shelf,  $(Q, \triangleleft)$  is a right shelf, and the following hold:

(iii) (**right inverse property**)  $(x \triangleright y) \triangleleft x = y$  for all  $x, y \in Q$ , and

(iv) (**left inverse property**)  $x \triangleright (y \triangleleft x) = y$  for all  $x, y \in Q$ .

That is, (iii) and (iv) simply say that  $x \triangleright -$  and  $- \triangleleft x$  are inverse operations. Once again, groups serve as the primary examples of racks, where the rack operations are left and right conjugation. Again, a routine computation gives the two inverse properties, which result from the fact that left and right conjugation are inverses.

Just as a rack is comprised of two shelves which fit together nicely, a quandle is made up of two ‘spindles’:

**Definition 55.** A left spindle  $(S, \triangleright)$  is a left shelf such that

(v) (**left idempotence**)  $x \triangleright x = x$  for all  $x \in S$ .

**Definition 56.** A right spindle  $(S, \triangleleft)$  is a right shelf such that

(vi) (**right idempotence**)  $x \triangleleft x = x$  for all  $x \in S$ .

As in the case of shelves, we will use the term **spindle** to refer to left spindles.

We now arrive at the notion of a ‘quandle’, which is a rack satisfying an additional axiom that is related to the first Reidemeister move:

**Definition 57.** A quandle  $(Q, \triangleright, \triangleleft)$  is a rack such that  $(Q, \triangleright)$  is a left spindle and  $(Q, \triangleleft)$  is a right spindle.

The motivating example of a quandle is a group with the operations of left and right conjugation. In their paper [FR], Fenn and Rourke mention that Conway and Wraith regarded a quandle as the wreckage left behind when we start with a group and discard the group operation so that only the notion of conjugacy remains. Indeed, this is why Fenn and Rourke use the term ‘rack’ to describe a structure still weaker than that of a quandle, as in the phrase “rack and ruin”.

Another example of a quandle is the **reflection quandle** whose underlying set is  $\mathbb{R}^2$  and has quandle operations defined as:

$$a \triangleright b = b \triangleleft a = \text{“reflect } b \text{ about } a”} = 2a - b.$$

Notice that left and right conjugation are the same in this case because reflection in the plane is its own inverse. Numerous examples of quandles and racks can be found in the work of Fenn and Rourke [FR].

In order to define the categories of these concepts, we must first define shelf, rack, spindle and quandle homomorphisms. Such morphisms must preserve the left and/or right conjugation.

**Definition 58.** Given quandles  $Q, Q'$ , a function  $f: Q \rightarrow Q'$  is a **quandle morphism** if  $f(x \triangleright y) = f(x) \triangleright f(y)$  and  $f(x \triangleleft y) = f(x) \triangleleft f(y)$ .

Shelf, rack and spindle morphisms are defined similarly. With these morphisms, we obtain the categories: **Shelf**, **Rack**, **Spind**, and **Quand**.

Due to the relationship between groups and quandles, we can describe a functor  $Conj: \text{Grp} \rightarrow \text{Quand}$  from the category of groups to the category of quandles, which sends any group  $G$  to  $(G, \triangleright, \triangleleft)$  where the operations are simply left and right conjugation. In addition, we can consider  $n$ -fold conjugation:  $x \triangleright y = x^n y x^{-n}$  and  $x \triangleleft y = y^{-n} x y^n$  and obtain another functor  $Conj_n: \text{Grp} \rightarrow \text{Quand}$  that is defined in the obvious way.

It is important to notice that not every quandle is of the form  $Conj(G)$  for some group  $G$ . In addition, since every quandle is a rack, we have natural inclusion of the category of quandles in the category of racks. However, the converse is not necessarily true. For example, the **cyclic rack of order  $n$**  is a rack but not a quandle. This rack consists of a set  $C_n = \{0, 1, \dots, n-1\}$  with the rack operations defined by  $x \triangleright y = (y+1) \bmod n$  and  $y \triangleleft x = (y-1) \bmod n$ .

Readers familiar with the notions of ‘rack’ and ‘quandle’ may recall these definitions and notation in another guise. Fenn and Rourke used the following, equivalent, definition:

**Definition 59.** A **quandle**  $(Q, \triangleleft)$  is a set  $Q$  equipped with a binary operation  $\triangleleft: Q \times Q \rightarrow Q$  defined as  $(a, b) \mapsto a \triangleleft b$  which satisfies:

- (a)  $x \triangleleft x = x$  for all  $x \in Q$
- (b) For all  $x, y \in Q$ , there exists a unique  $z \in Q$  such that  $z \triangleleft x = y$ .
- (c)  $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$  for all  $x, y, z \in Q$ .

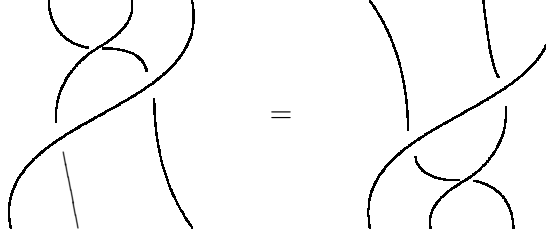
The second condition in the above definition is equivalent to the existence of a binary operation  $\triangleright: Q \times Q \rightarrow Q$  such that the two operations  $\triangleright$  and  $\triangleleft$  satisfy the two inverse properties of Definition 54. While they used the above definition, Fenn and Rourke chose to use exponential notation for their operation, since the operation is not symmetric. That is, whereas we used  $x \triangleleft y$  to represent the result of  $y$  acting on  $x$  from the right, Fenn and Rourke notated this same quantity as  $x^y$ . While the exponential notation certainly has its advantages, we have chosen to use the notation  $x \triangleright y$  and  $x \triangleleft y$  because this language makes it easier to describe generators and relations and will prove easier to categorify in the future. That is, when we define 2-quandles, which will be categories equipped with two conjugation *functors*, we can continue to use the notation  $\triangleright$  and  $\triangleleft$  for these functors.

We continue by exhibiting the correlation between shelves, racks, spindles, and quandles and the theory of braids. In particular, we illustrate the relationship between the Reidemeister moves and the axioms of distributivity, inverses, and idempotence.

### 3.1.2 Relation to Topology

The Yang–Baxter equation arises in many contexts in mathematics and physics. All these concepts are related by the fact that this equation is an algebraic distillation of

the ‘third Reidemeister move’ in knot theory:



The Yang–Baxter equation makes sense in any monoidal category. Originally mathematical physicists concentrated on solutions in the category of *vector spaces* with its usual tensor product, obtaining solutions from quantum groups. More recently there has been interest in solutions of the Yang–Baxter equation in the category of *sets*, with its Cartesian product [G-I, GV, ESS, EGS, H, LYZ, O, R, S, WX]. In this section we prove that any shelf (and thus any rack, spindle, or quandle) gives a solution to the Yang–Baxter equation in the category of sets. In particular, this implies that any group gives a solution to the Yang–Baxter equation in which the crossing:



is related to the *action of conjugating one group element by another*. Similarly, we prove that any Lie algebra gives a solution to the Yang–Baxter equation in the category of vector spaces, in which the crossing is related to the *bracket operation*.

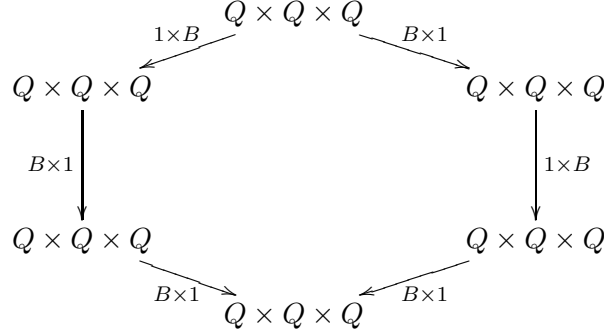
These two results illustrate one of the connections between Lie algebras and shelves, by demonstrating that the Jacobi identity and self-distributivity axiom are, in a suitable context, equivalent. Not surprisingly, our results for shelves and Lie algebras are closely related! We will make this relation much more explicit in Section 3.2.4, where we show how spindles in a certain category  $\mathcal{C}$  come from Lie groups, and give Lie algebras. Furthermore, we show that the first and second Reidemeister moves in knot theory are related to the idempotence and inverse properties.

We begin by recalling the Yang–Baxter equation:

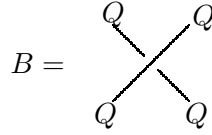
**Definition 60.** *Given a set  $Q$  and an isomorphism  $B: Q \times Q \rightarrow Q \times Q$ , we say  $B$  is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:*

$$(B \times 1)(1 \times B)(B \times 1) = (1 \times B)(B \times 1)(1 \times B),$$

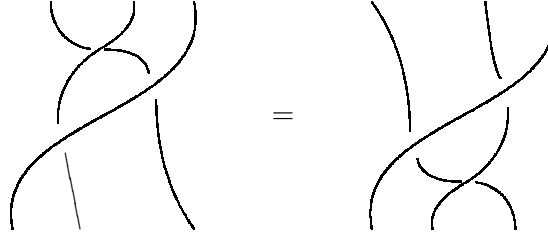
or in other words, that this diagram commutes:



If we draw  $B: Q \times Q \rightarrow Q \times Q$  as a braiding:



the Yang–Baxter equation says that:

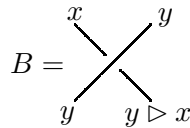


This is called the ‘third Reidemeister move’ in knot theory [BZ], and it gives the most important relations in Artin’s presentation of the braid group [Bir]. As a result, any solution of the Yang–Baxter equation gives an invariant of braids.

In general, a ‘braiding’ operation provides a diagrammatic description of the process of switching the order of two things. This idea is formalized in the concept of a braided monoidal category, where the braiding is an isomorphism

$$B_{x,y}: x \otimes y \rightarrow y \otimes x.$$

To show that the distributive law in the definition of a shelf is equivalent to the Yang–Baxter equation, we define a braiding operation in terms of the conjugation operation. Given a set  $Q$  and binary operation  $\triangleright: Q \times Q \rightarrow Q$ , we can define a braiding map  $B: Q \times Q \rightarrow Q \times Q$  as  $B(x, y) = (y, y \triangleright x)$ . That is, since we have switched the order of  $x$  and  $y$ , we introduce a ‘correction’ term involving the conjugation operation. We draw  $B$  as a positive crossing:



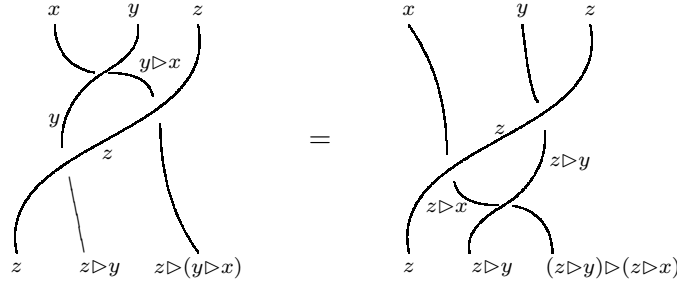
Since the third Reidemeister move involves three strands and the shelf distributive law involves three shelf elements, it should not come as a surprise that the Yang–Baxter equation is actually *equivalent* to the shelf distributive law is actually equivalent in a suitable context:

**Lemma 61.** *Let  $Q$  be a set equipped with a binary operation  $\triangleright: Q \times Q \rightarrow Q$ . The braiding operation,  $B$ , defined above satisfies the Yang–Baxter equation:*

$$(B \times 1)(1 \times B)(B \times 1) = (1 \times B)(B \times 1)(1 \times B)$$

*if and only if  $(Q, \triangleright)$  is a left shelf.*

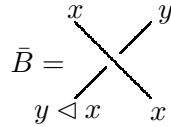
**Proof.** The braiding  $B$  satisfies the Yang–Baxter equation if and only if the following braid equation holds:



That is,  $B$  satisfies the Yang–Baxter equation if and only if  $z \triangleright (y \triangleright x) = (z \triangleright y) \triangleright (z \triangleright x)$ , which is the left distributive law, (i). Hence, the braiding gives a solution of the Yang–Baxter equation if and only if  $(Q, \triangleright)$  is a left shelf.  $\square$

Thus, left shelves, and hence left conjugation, correspond to ‘positive’ or right-handed crossings. While this may seem perverse, it has the effect that our ‘favorite’ conjugation  $x \triangleright y = xyx^{-1}$  corresponds to the topologist’s ‘favorite’ crossing, namely the right-handed one. Recall that our ‘favorite’ conjugation is the one that differentiates to give  $[x, y]$  in a Lie algebra. Analogously, right shelves and right conjugation correspond to ‘negative’ or left-handed crossings. We now show that we can define a Yang–Baxter operator on right shelves as well.

Given a set  $Q$  and binary operation  $\triangleleft: Q \times Q \rightarrow Q$ , we may define a new braiding map  $\bar{B}$  as  $\bar{B}(x, y) = (y \triangleleft x, x)$ . We draw  $\bar{B}$  as a negative crossing:

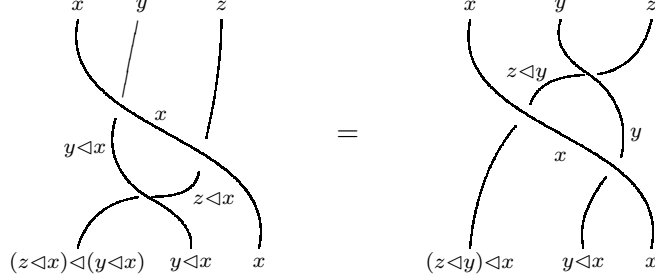


Notice that in the illustrations of both  $B$  and  $\bar{B}$ , the strand above always acts on the strand below.

Analogous to the comment made above, since the right distributive law involves three right shelf elements, it, too, is equivalent to the Yang–Baxter equation in an appropriate context:

**Lemma 62.** *Let  $Q$  be a set equipped with a binary operation  $\triangleleft: Q \times Q \rightarrow Q$ . The braiding map  $\bar{B}$  satisfies the Yang–Baxter equation if and only if  $(Q, \triangleleft)$  is a right shelf.*

**Proof.** The braiding  $\bar{B}$  satisfies the Yang-Baxter equation if and only if the following braid equation holds:



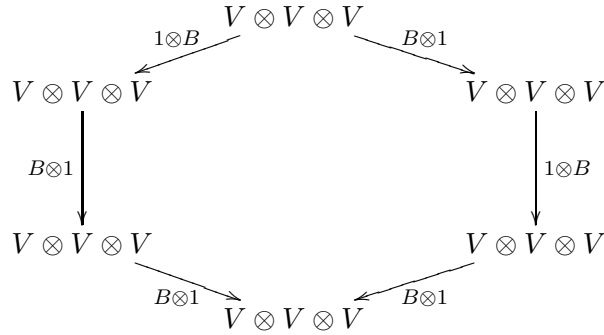
Thus,  $\bar{B}$  satisfies the Yang-Baxter equation if and only if  $((z \triangleleft y) \triangleleft x, y \triangleleft x, x) = ((z \triangleleft x) \triangleleft (y \triangleleft x), y \triangleleft x, x)$ , which is the right distributive law, (ii). Hence, the braiding  $\bar{B}$  gives a solution of the Yang-Baxter equation if and only if  $(Q, \triangleright)$  is a right shelf.  $\square$

Therefore, shelves give set-theoretic solutions of the Yang-Baxter equation. Of course, since racks, spindles, and quandles are shelves equipped with some combination of extra structure and extra properties, they will also give solutions to the Yang-Baxter equation. We proceed by showing that Lie algebras also give solutions of the Yang-Baxter equation, a result due to James Dolan [D]. Now, we consider vector spaces:

**Definition 63.** Given a vector space  $V$  and an isomorphism  $B: V \otimes V \rightarrow V \otimes V$ , we say  $B$  is a **Yang-Baxter operator** if it satisfies the **Yang-Baxter equation**, which says that:

$$(B \otimes 1)(1 \otimes B)(B \otimes 1) = (1 \otimes B)(B \otimes 1)(1 \otimes B),$$

or in other words, that this diagram commutes:



As in the set-theoretic definition, if we draw  $B$  as a braiding, the Yang-Baxter equation is equivalent to the third Reidemeister move.

We have seen that the braiding defined on a shelf  $Q$  formalized the failure of  $x$  and  $y$  to commute by introducing a correction term involving the conjugation operation. Since the bracket  $[x, y]$  in a Lie algebra measures the difference between  $xy$  and  $yx$ , it should not be too surprising that we can get a Yang-Baxter operator from any Lie algebra. Here, when we switch the order of two Lie algebra elements  $x$  and  $y$ , we add on a correction term involving the bracket:  $[x, y]$ . And, just as we have already observed for shelves, since the

third Reidemeister move involves three strands, while the Jacobi identity involves three Lie algebra elements, it should also not be surprising that the Yang–Baxter equation is actually *equivalent* to the Jacobi identity in a suitable context:

**Proposition 64.** *Let  $L$  be a vector space equipped with a skew-symmetric bilinear operation  $[\cdot, \cdot]: L \times L \rightarrow L$ . Let  $L' = k \oplus L$  and define the isomorphism  $B: L' \otimes L' \rightarrow L' \otimes L'$  by*

$$B((a, x) \otimes (b, y)) = (b, y) \otimes (a, x) + (1, 0) \otimes (0, [x, y]).$$

*Then  $B$  is a solution of the Yang–Baxter equation if and only if  $[\cdot, \cdot]$  satisfies the Jacobi identity.*

**Proof.** Applying the left-hand side of the Yang–Baxter equation to an object  $(a, x) \otimes (b, y) \otimes (c, z)$  of  $L' \otimes L' \otimes L'$  yields:

$$\begin{aligned} (a, x) \otimes (b, y) \otimes (c, z) &\longmapsto (b, y) \otimes (a, x) \otimes (c, z) + (1, 0) \otimes (0, [x, y]) \otimes (c, z) \\ &\longmapsto (b, y) \otimes (c, z) \otimes (a, x) + (b, y) \otimes (1, 0) \otimes (0, [x, z]) \\ &\quad + (1, 0) \otimes (c, z) \otimes (0, [x, y]) + (1, 0) \otimes (1, 0) \otimes (0, [[x, y], z]) \\ &\longmapsto (c, z) \otimes (b, y) \otimes (a, x) + (1, 0) \otimes (0, [y, z]) \otimes (a, x) \\ &\quad + (1, 0) \otimes (b, y) \otimes (0, [x, z]) + (c, z) \otimes (1, 0) \otimes (0, [x, y]) \\ &\quad + (1, 0) \otimes (1, 0) \otimes (0, [[x, y], z]) \end{aligned}$$

while applying the right-hand side produces:

$$\begin{aligned} (a, x) \otimes (b, y) \otimes (c, z) &\longmapsto (a, x) \otimes (c, z) \otimes (b, y) + (a, x) \otimes (1, 0) \otimes (0, [y, z]) \\ &\longmapsto (c, z) \otimes (a, x) \otimes (b, y) + (1, 0) \otimes (0, [x, z]) \otimes (b, y) \\ &\quad + (1, 0) \otimes (a, x) \otimes (0, [y, z]) \\ &\longmapsto (c, z) \otimes (b, y) \otimes (a, x) + (c, z) \otimes (1, 0) \otimes (0, [x, y]) \\ &\quad + (1, 0) \otimes (b, y) \otimes (0, [x, z]) + (1, 0) \otimes (1, 0) \otimes (0, [[x, z], y]) \\ &\quad + (1, 0) \otimes (0, [y, z]) \otimes (a, x) + (1, 0) \otimes (1, 0) \otimes (0, [x, [y, z]]) \end{aligned}$$

Notice that both sides consist of the same four uninteresting terms. The remaining terms are equal if and only if the Jacobi identity is satisfied in  $L$ .  $\square$

Now that we have seen that the shelf distributive law and Jacobi identity are equivalent to the third Reidemeister move, it is natural to wonder whether Lie algebras actually *are* shelves. It turns out that they are! Actually, they are a special sort of spindle in a certain category. Of course, it has to be a category with products for the notion of spindle to make sense. Furthermore, while forming the space  $k \oplus L$  allowed us to get a solution of the Yang–Baxter equation, this space is not quite large enough to enable us to define a spindle structure on it. Therefore, we need something a bit larger and fancier to get this special sort of spindle from a Lie algebra. We will see that we need a space more like the symmetric algebra,  $SL$ , when we return to this idea and make it more precise in Section 3.2.4. For now, we continue by showing the relationship between the second Reidemeister move and the two inverse properties.



The definitions given in Section 3.1.1 show that the six quandle axioms: left and right self-distributivity, left and right inverse laws, and left and right idempotence are completely symmetrical. In fact, it turns out that two of these six axioms are implied by the other four! We will show that while we may omit one of the two self-distributive laws and one of the two idempotence laws in the definition of a quandle, we may *not* eliminate either of the two inverse properties. We will use the braidings  $B$  and  $\bar{B}$  described above to demonstrate that we may omit one of the two self-distributive laws from the definition of a rack, and therefore from the definition of a quandle, without harm. Specifically we will prove that a shelf  $(Q, \triangleright)$  equipped with an additional operation  $\triangleleft: Q \times Q \rightarrow Q$  satisfying the two inverse properties (iii) and (iv) is actually a rack. We begin by showing that  $B$  and  $\bar{B}$  are inverses when  $(Q, \triangleright, \triangleleft)$  satisfies the two inverse properties.

**Lemma 65.** *Let  $Q$  be a set equipped with two binary operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$ . Then the braiding operations  $B$  and  $\bar{B}$ ,*

$$B = \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ y \quad x \triangleright y \end{array} \qquad \bar{B} = \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \\ y \triangleleft x \quad x \end{array}$$

*are inverses if and only if  $(Q, \triangleright, \triangleleft)$  satisfies the two inverse properties.*

**Proof.** The braidings  $B$  and  $\bar{B}$  are inverses if and only if the following braid equation holds:

$$\begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ y \triangleleft x \quad x \\ \diagup \quad \diagdown \\ x \quad x \triangleright (y \triangleleft x) \end{array} = \begin{array}{c} x \quad y \\ | \quad | \\ x \quad y \end{array} = \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \\ y \quad y \triangleright x \\ \diagdown \quad \diagup \\ (y \triangleright x) \triangleleft y \quad y \end{array}$$

Thus,  $B$  and  $\bar{B}$  are inverses if and only if  $x \triangleright (y \triangleleft x) = y$  and  $(y \triangleright x) \triangleleft y = x$ , which are the two inverse properties, (iii) and (iv). Therefore,  $B$  and  $\bar{B}$  are inverses if and only if  $(Q, \triangleright, \triangleleft)$  satisfies these two properties.  $\square$

When  $(Q, \triangleright)$  is a left shelf and  $(Q, \triangleleft)$  is a right shelf, we have an immediate corollary:

**Corollary 66.** *Let  $Q$  be a set equipped with two binary operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$  such that  $(Q, \triangleright)$  is a left shelf and  $(Q, \triangleleft)$  is a right shelf. Then the braiding operations  $B$  and  $\bar{B}$  are inverses if and only if  $(Q, \triangleright, \triangleleft)$  is a rack.*

Henceforth we will write  $B^{-1}$  for  $\bar{B}$ . It is easy to see that if  $B$  satisfies the Yang–Baxter equation, then so does  $B^{-1}$ :

**Lemma 67.** *If  $B: Q \times Q \rightarrow Q \times Q$  is invertible and satisfies the Yang–Baxter equation, then  $B^{-1}$  does as well.*

**Proof.** We begin with the Yang–Baxter equation satisfied by  $B$  and apply inverses to both sides to see that  $B^{-1}$  also satisfies this equation:

$$(1 \times B^{-1})(B^{-1} \times 1)(1 \times B^{-1}) = (B^{-1} \times 1)(1 \times B^{-1})(B^{-1} \times 1) \quad \square$$

As a corollary to Lemma 65, note that if  $(Q, \triangleright)$  is a left shelf and the braiding  $B$  defined on it is invertible, then  $B^{-1}$  satisfies the Yang–Baxter equation by Lemma 67. Therefore we can use  $B^{-1}$  to *define* an operation  $\triangleleft: Q \times Q \rightarrow Q$  as  $\triangleleft(x, y) = x \triangleleft y = B^{-1}\pi_1(x, y)$ , where we have expressed the composite in the non-traditional sense. It is not difficult to show that with this definition,  $(Q, \triangleleft)$  is a right shelf, and thus  $(Q, \triangleright, \triangleleft)$  is a rack.

We finally arrive at our desired conclusion: A shelf together with an additional operation satisfying the two inverse properties,  $x \triangleright (y \triangleleft x) = y = (x \triangleright y) \triangleleft x$ , is a rack. That is, the right distributive law is implied by the left distributive law together with the two inverse properties.

**Proposition 68.** *A left shelf  $(Q, \triangleright)$  together with an operation  $\triangleleft: Q \times Q \rightarrow Q$  that satisfies the inverse properties, (iii) and (iv), also satisfies the right distributive law, (ii). That is,  $(Q, \triangleright, \triangleleft)$  is a rack.*

**Proof.** Define  $B: Q \times Q \rightarrow Q \times Q$  in terms of  $\triangleright$  as before. The braiding  $B$  satisfies the Yang–Baxter equation by Lemma 61 since  $(Q, \triangleright)$  is a left shelf. By hypothesis,  $(Q, \triangleright, \triangleleft)$  satisfies the two inverse properties, so Lemma 65 establishes that  $B$  is invertible. Since  $B$  is invertible and satisfies the Yang–Baxter equation, Lemma 67 demonstrates that  $B^{-1}$  also satisfies the Yang–Baxter equation. Finally, Lemma 62 shows that  $(Q, \triangleright)$  is a right shelf, so that  $(Q, \triangleright, \triangleleft)$  is a rack.  $\square$

Notice that we could have similarly shown that a right shelf  $(Q, \triangleleft)$  together with an operation  $\triangleright: Q \times Q \rightarrow Q$  that satisfies the inverse properties also satisfies the left distributive law, (i). Therefore, we see that either the right distributive law, (ii), or the left distributive law, (i), is superfluous when defining a rack. Furthermore, we can omit either the right idempotence law, (vi), or the left idempotence law, (v), in the definition of a quandle because either one implies the other with the help of the inverse properties. For instance, assuming  $x \triangleright x = x$ , we have:

$$x \triangleleft x = (x \triangleright x) \triangleleft x = x,$$

where the first equality holds by the idempotence law (v) and the second follows from the inverse property (iii). We use the other inverse property to show that  $x \triangleleft x = x$  implies  $x \triangleright x = x$ . This result together with Lemma 68 illustrates the power and necessity of *both* inverse properties. That is, we have taken advantage of the connection between algebra and topology to show that we are unable to eliminate either of the two inverse axioms from the definition of a rack. This should not be terribly surprising since one of the inverse laws says that  $x \triangleright -$  is a left inverse of  $- \triangleleft x$ , while the other says that it is a right inverse.

We conclude this section by illustrating the relation between the first Reidemeister move and the idempotence conditions. However, because the first Reidemeister move

involves ‘framed braids’, which are more general than braids, we are unable to apply our braiding and inverse braiding maps to this move. Therefore, we make the following conventions:

$$\begin{array}{c} x \\ \text{loop on left} \\ x < x \end{array} = \begin{array}{c} x \\ \text{straight} \\ x \end{array} = \begin{array}{c} x \\ \text{loop on right} \\ x > x \end{array}$$

We continue to illustrate the relationship of shelves, racks, spindles, and quandles to topology by showing how these structures give invariants of different sorts of braids.

### 3.1.3 Braid and Framed Braid Groups and Monoids

Thus far we have demonstrated the relationship between the quandle axioms and the Reidemeister moves. We would now like to use that relation to illustrate a stronger connection between these algebraic structures and certain collections of braids. For reasons that are still somewhat mysterious, we will actually be more interested in ‘quasi-idempotent’ shelves, which are shelves satisfying an additional property:

**Definition 69.** *A shelf  $(Q, \triangleright)$  is **left quasi-idempotent** if it satisfies*

(vii) (**left quasi-idempotence**)  $(x \triangleright x) \triangleright y = x \triangleright y$  for all  $x, y \in Q$ .

Of course, right shelves may also be quasi-idempotent and thus satisfy  $y \triangleleft (x \triangleleft x) = y \triangleleft x$ . We remark that quandles are trivially left and right quasi-idempotent, and will show that racks are quasi-idempotent as well. Moreover, the rack axioms imply other useful properties such as mixed distributive laws:

**Lemma 70.** *Given a rack  $(Q, \triangleright, \triangleleft)$ , the following additional axioms hold:*

- (1) (**left quasi-idempotence**)  $(x \triangleright x) \triangleright y = x \triangleright y$
- (2) (**right quasi-idempotence**)  $y \triangleleft (x \triangleleft x) = y \triangleleft x$
- (3) (**mixed distributive law**)  $x \triangleright (y \triangleleft z) = (x \triangleright y) \triangleleft (x \triangleright z)$
- (4) (**mixed distributive law**)  $(x \triangleright y) \triangleleft z = (x \triangleleft z) \triangleright (y \triangleleft z)$

**Proof.** The mixed distributive laws imply the quasi-idempotence laws, so we begin by verifying (3) and (4). In order to simplify the proofs, we will use the fact that

$$x \triangleright (y \triangleleft y) = A \Leftrightarrow [x \triangleright (y \triangleleft y)] \triangleleft x = A \triangleleft x \Leftrightarrow y \triangleleft y = A \triangleleft x,$$

which holds by the left inverse property of racks.

Using this fact, demonstrating (3) is equivalent to showing

$$[x \triangleright (y \triangleleft z)] \triangleleft x = [(x \triangleright y) \triangleleft (x \triangleright z)] \triangleleft x \Leftrightarrow y \triangleleft z = [(x \triangleright y) \triangleleft (x \triangleright z)] \triangleleft x.$$

But,

$$[(x \triangleright y) \triangleleft (x \triangleright z)] \triangleleft x = [(x \triangleright y) \triangleleft x] \triangleleft [(x \triangleright z) \triangleleft x] = (y \triangleleft z)$$

where the first equality holds by the self distributivity law, (ii), and the second equality follows from two applications of the right inverse law, (iii). Property (4) is proved in a completely analogous way using the other self distributivity law, (i), and the left inverse law, (iv).

Again, using the above fact, property (1) is equivalent to

$$[(x \triangleright x) \triangleright y] \triangleleft x = (x \triangleright y) \triangleleft x \quad \Leftrightarrow \quad [(x \triangleright x) \triangleright y] \triangleleft x = y$$

using the right inverse property. But,

$$[(x \triangleright x) \triangleright y] \triangleleft x = [(x \triangleright x) \triangleleft x] \triangleright (y \triangleleft x) = x \triangleright (y \triangleleft x) = y$$

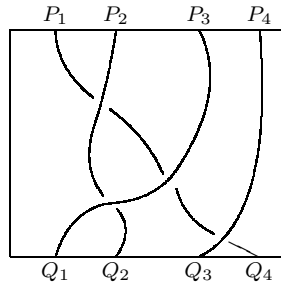
where the first equality results from property (4) above, the second equality holds by applying the right inverse property to the first term, and the final equality follows from the left inverse property. Thus, (1) holds. Property (2) is proved analogously, using (3) above.  $\square$

As suggested by the name, idempotence is stronger than quasi-idempotence. This follows from the fact that there exist quandles which are not racks, such as the cyclic rack described in the Section 3.1.1. We now turn to the task of showing how the four categories of these algebraic structures are related to certain braid groups.

In 1925 E. Artin presented an algebraic description of braids. We begin by reminding the reader of both the geometric and algebraic descriptions of the braid group.

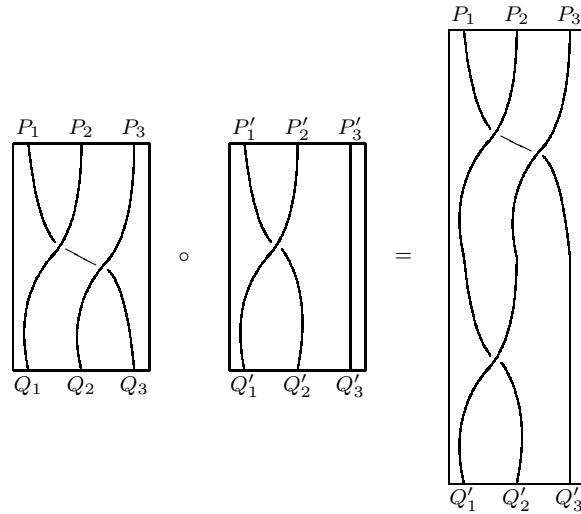
**Definition 71.** *Let  $R$  be a rectangle in 3-space. On opposite sides of  $R$ , label equidistant points  $P_i$  and  $Q_i$ , where  $1 \leq i \leq n$ . Then, let  $f_i$ ,  $1 \leq i \leq n$  be  $n$  simple disjoint smooth arcs in  $\mathbb{R}^3$  which begin at  $P_i$  and end at  $Q_{\tau(i)}$  where  $i \mapsto \tau(i)$  is a permutation on  $\{1, 2, \dots, n\}$ . We require that each  $f_i$  head downward as we move along any one of these  $f_i$  from the top of  $R$  to the bottom. That is, each  $f_i$  intersects any horizontal plane between the top and bottom of  $R$  exactly once. We say that the set of  $n$  arcs  $f_i$  constitute an **n-braid**.*

As an example:

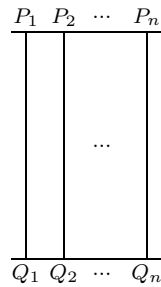


is a 4-braid. Two braids are said to be **equivalent** if they are ambient isotopic via an isotopy that fixes the boundary points  $P_i$  and  $Q_i$ .

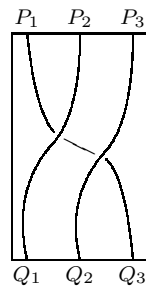
Given two  $n$ -strand braids  $b$  and  $b'$  we may multiply, or compose, them by identifying the terminal points  $Q_i$  of  $b$  with the initial points  $P'_i$  of  $b'$ . That is, we stack  $b$  on top of  $b'$ :



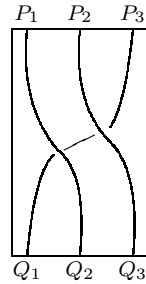
to obtain the product. There exists an obvious identity with respect to this multiplication, namely the braid with no crossings:



In addition, each braid  $b$

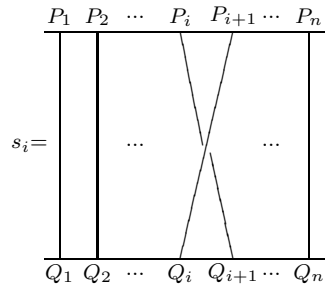


has an inverse with respect to this multiplication,  $b^{-1}$

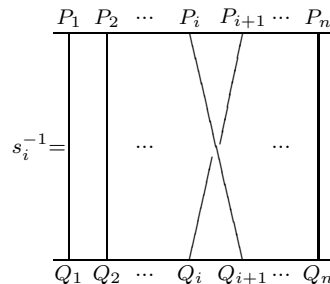


obtained by reflecting  $b$  in a plane perpendicular to it. The collection of  $n$ -braids together with the multiplication defined above describes a group known as the **braid group**,  $B_n$ . To be precise, an element of this group is a braid  $b$  together with all the braids equivalent to it. That is, an element of  $B_n$  is an equivalence class of braids,  $[b]$ .

Thus far we have described the braid group from a purely topological viewpoint. We can also give an algebraic description of braids by listing which of the strands cross over and under one another as we move down the braid. We may always arrange our braid so that no two crossings occur at exactly the same height. We will use  $s_i$  to denote the  $(i+1)^{st}$  strand crossing over the  $i^{th}$  strand:



and  $s_i^{-1}$  to denote the  $i^{th}$  strand crossing over the  $(i+1)^{st}$  strand:



Using these two types of crossings, we may describe the braid group in terms of generators and relations. This presentation of the braid group takes the following form:

**Proposition 72.** *The braid group,  $B_n$ , may be presented as*

$$B_n = \langle s_1, \dots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i \leq n-2), \quad s_i s_j = s_j s_i \quad (|i-j| > 1) \rangle.$$

**Proof.** This was proved by Artin [A]; a convenient reference is the book by Birman [Bir].  
□

Notice that the first relation, when drawn, is the third Reidemeister move. The second relation indicates that when two pairs of strands are far enough apart,  $|i - j| > 1$ , the order in which we braid them is irrelevant.

When we allow our strands to rotate, we obtain ‘framed braids’:

**Definition 73.** Let  $D^2$  denote the closed unit ball in  $\mathbb{R}^2$ . Let  $e_i: D^2 \rightarrow \mathbb{R}^2$ ,  $1 \leq i \leq n$ , be disjoint oriented balls embedded in  $\mathbb{R}^2$ . Then an **n-strand framed braid** is an oriented embedding  $F$  of the disjoint union of  $n$  solid cylinders  $[0, 1] \times D^2$  in  $[0, 1] \times \mathbb{R}^2$ , such that  $F_i(0, \cdot) = F_i(1, \cdot) = e_{\sigma(i)}$ , for some  $\sigma \in S_n$ , where  $F_i$  denotes the embedding of the  $i$ th cylinder, and such that  $F_i(t, x) = (t, F_{i,t}(x))$  for some function  $F_{i,t}: D^2 \rightarrow \mathbb{R}^2$ .

The set of homotopy classes of  $n$ -strand framed braids, where the homotopy is required to preserve the above conditions on  $F$ , is denoted by  $FB_n$ . This becomes a group in an obvious way, called the **framed braid group**. The framed braid group  $FB_n$  has the braid group  $B_n$  as a quotient group in an obvious way, so there is a quotient map

$$\pi: FB_n \rightarrow B_n.$$

The framed braid group keeps track of both the interchange of strands as well as their rotations. The rotations in the framed braid group correspond to  $2\pi$  rotations of the  $i$ th strand which will be drawn as tangles:



As in the case of the braid group, the framed braid group can be described in terms of generators and relations. In addition to the generators  $s_i$  and  $s_i^{-1}$  and the relations they satisfy in the braid group, the framed braid group has tangles  $t_i$  and  $t_i^{-1}$ ,  $1 \leq i \leq n$ , and relations:

$$\begin{aligned} t_i t_j &= t_j t_i \\ t_i s_j &= s_j t_i \quad \text{if } i \geq j + 2 \text{ or } i < j \\ t_{i+1} s_i &= s_i t_i \\ t_i s_i &= s_i t_{i+1}, \end{aligned}$$

These relations tell us how the braidings and twists interact with one another. We clearly have a natural inclusion:

$$\iota: B_n \hookrightarrow FB_n$$

such that  $\iota\pi$  is the identity on  $B_n$ , where  $\pi$  is the quotient map mentioned above that sends the generators  $t_i$  to 1.

We can also consider the monoids generated by  $s_i$  and  $t_i$  without their inverses. The **positive braid monoid**  $B_n^+$  is the monoid generated by the right-handed braidings,  $s_i$ . The **positive framed braid monoid**  $FB_n^+$  is the monoid generated by the right-handed braidings,  $s_i$ , and the right-handed twists,  $t_i$ . We have the following commutative diagram of monoid homomorphisms:

$$\begin{array}{ccc} FB_n^+ & \longrightarrow & B_n^+ \\ \downarrow & & \downarrow \\ FB_n & \longrightarrow & B_n \end{array}$$

We have a similar square illustrating the forgetful functors between the categories of the algebraic structures described in Section 3.1.1:

$$\begin{array}{ccc} \text{Shelf} & \longleftarrow & \text{Spind} \\ \uparrow & & \uparrow \\ \text{Rack} & \longleftarrow & \text{Quand} \end{array}$$

When considered together, these squares demonstrate a contravariant relationship between the braid and framed braid groups and monoids and our four categories:

**Theorem 74.** *Given a set  $Q$  equipped with operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$ , and defining maps  $s_i, t_i, s_i^{-1}$ , and  $t_i^{-1}$  on  $Q^n$  as*

- $s_i: (q_1, q_2, \dots, q_i, \dots, q_n) \mapsto (q_1, q_2, \dots, q_{i-1}, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n)$
- $t_i: (q_1, q_2, \dots, q_i, \dots, q_n) \mapsto (q_1, q_2, \dots, q_{i-1}, q_i \triangleright q_i, q_{i+1}, \dots, q_n)$
- $s_i^{-1}: (q_1, q_2, \dots, q_i, \dots, q_n) \mapsto (q_1, q_2, \dots, q_{i-1}, q_{i+1} \triangleleft q_i, q_i, q_{i+2}, \dots, q_n)$
- $t_i^{-1}: (q_1, q_2, \dots, q_i, \dots, q_n) \mapsto (q_1, q_2, \dots, q_{i-1}, q_i \triangleleft q_i, q_{i+1}, \dots, q_n)$

these maps give an action of  $\begin{cases} FB_n^+ \\ B_n^+ \\ FB_n \\ B_n \end{cases}$  on  $Q^n$  if and only if  $\begin{cases} (Q, \triangleright) \text{ is a quasi-idempotent shelf} \\ (Q, \triangleright) \text{ is a spindle} \\ (Q, \triangleright, \triangleleft) \text{ is a rack} \\ (Q, \triangleright, \triangleleft) \text{ is a quandle} \end{cases}$

**Proof.** We begin by proving that if



$$\begin{cases} (Q, \triangleright) \text{ is a quasi-idempotent shelf} \\ (Q, \triangleright) \text{ is a spindle} \\ (Q, \triangleright, \triangleleft) \text{ is a rack} \\ (Q, \triangleright, \triangleleft) \text{ is a quandle} \end{cases}$$

then we obtain an action of

$$\begin{cases} FB_n^+ \\ B_n^+ \\ FB_n \\ B_n \end{cases}$$

on  $Q^n$  where the generators  $s_i, t_i, s_i^{-1}$ , and  $t_i^{-1}$  act as above. It suffices for each case to show that the relations in the appropriate collection of braids are satisfied.

We first consider the case when  $(Q, \triangleright)$  is a quasi-idempotent shelf. Therefore, we must show that the relations in the positive framed braid monoid hold. We begin by examining the relations in the braid group:

(a) Let  $q_i \in Q$ . Assume  $i < j$  and  $|i - j| > 1$ . Then,

$$\begin{aligned} s_i s_j(q_1, \dots, q_n) &= s_i(q_1, q_2, \dots, q_{j+1}, q_{j+1} \triangleright q_j, q_{j+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_{j+1}, q_{j+1} \triangleright q_j, q_{j+2}, \dots, q_n) \\ &= s_j s_i(q_1, q_2, \dots, q_n). \end{aligned}$$

(b) Assume  $1 \leq i \leq n - 2$ . Then,

$$\begin{aligned} s_i s_{i+1} s_i(q_1, \dots, q_n) &= s_i s_{i+1}(q_1, q_2, \dots, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n) \\ &= s_i(q_1, q_2, \dots, q_{i+1}, q_{i+2}, q_{i+2} \triangleright (q_{i+1} \triangleright q_i), q_{i+3}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i+2}, q_{i+2} \triangleright q_{i+1}, q_{i+2} \triangleright (q_{i+1} \triangleright q_i), q_{i+3}, \dots, q_n), \end{aligned}$$

while on the other hand,

$$\begin{aligned} s_{i+1} s_i s_{i+1}(q_1, \dots, q_n) &= s_{i+1} s_i(q_1, \dots, q_i, q_{i+2}, q_{i+2} \triangleright q_{i+1}, \dots, q_n) \\ &= s_{i+1}(q_1, \dots, q_{i+2}, q_{i+2} \triangleright q_i, q_{i+2} \triangleright q_{i+1}, \dots, q_n) \\ &= (q_1, \dots, q_{i+2}, q_{i+2} \triangleright q_{i+1}, (q_{i+2} \triangleright q_{i+1}) \triangleright (q_{i+2} \triangleright q_i), \dots, q_n). \end{aligned}$$

The final two expressions are equal by the left distributive law, (i), in the definition of a shelf. We continue by verifying the relations involving the generators  $t_i$ :

(c) Assume without loss of generality that  $i < j$ . We have

$$\begin{aligned} t_i t_j(q_1, q_2, \dots, q_n) &= t_i(q_1, q_2, \dots, q_{j-1}, q_j \triangleright q_j, q_{j+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, q_i \triangleright q_i, q_{i+1}, \dots, q_{j-1}, q_j \triangleright q_j, q_{j+1}, \dots, q_n) \\ &= t_j t_i(q_1, q_2, \dots, q_n). \end{aligned}$$

We now consider the relations which describe how the braiding and twisting may interact.

(d) Let  $i \geq j + 2$ . Then,

$$\begin{aligned} t_i s_j(q_1, q_2, \dots, q_n) &= t_i(q_1, q_2, \dots, q_{j+1}, q_{j+1} \triangleright q_j, q_{j+2}, \dots, q_i, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{j+1}, q_{j+1} \triangleright q_j, q_{j+2}, \dots, q_i \triangleright q_i, \dots, q_n) \end{aligned}$$

while

$$\begin{aligned} s_j t_i(q_1, q_2, \dots, q_n) &= s_j(q_1, q_2, \dots, q_j, \dots, q_i \triangleright q_i, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{j+1}, q_{j+1} \triangleright q_j, q_{j+2}, \dots, q_i \triangleright q_i, \dots, q_n) \end{aligned}$$

We arrive at the same conclusion when  $i < j$ , since  $i$  and  $j$  are sufficiently far enough apart so as to not interfere with one another.

(e) Continuing on, we have

$$\begin{aligned} t_{i+1} s_i(q_1, q_2, \dots, q_n) &= t_{i+1}(q_1, q_2, \dots, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i+1}, (q_{i+1} \triangleright q_i) \triangleright (q_{i+1} \triangleright q_i), q_{i+2}, \dots, q_n), \end{aligned}$$

while on the other hand,

$$\begin{aligned} s_i t_{i+1}(q_1, q_2, \dots, q_n) &= s_i(q_1, q_2, \dots, q_{i-1}, q_i \triangleright q_i, q_{i+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, q_{i+1}, q_{i+1} \triangleright (q_i \triangleright q_i), q_{i+2}, \dots, q_n). \end{aligned}$$

These statements are equal by the right-distributive law, (i) satisfied by a shelf.

(f) Finally,

$$\begin{aligned} t_i s_i(q_1, q_2, \dots, q_n) &= t_i(q_1, q_2, \dots, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i+1} \triangleright q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n) \end{aligned}$$

while

$$\begin{aligned} s_i t_{i+1}(q_1, q_2, \dots, q_n) &= s_i(q_1, q_2, \dots, q_i, q_{i+1} \triangleright q_{i+1}, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i+1} \triangleright q_{i+1}, (q_{i+1} \triangleright q_{i+1}) \triangleright q_i, q_{i+2}, \dots, q_n). \end{aligned}$$

These quantities are equal because of the quasi-idempotence of our shelf.

We next consider the case when  $Q$  is a spindle. We need to verify the relations of the positive braid monoid, and that the map  $t_i$  acts as the identity. The two relations of the positive braid monoid have already been demonstrated in (a) and (b) above. Therefore, we need only to examine the action of  $t_i$ :

(g)

$$\begin{aligned} t_i(q_1, q_2, \dots, q_i, \dots, q_n) &= (q_1, q_2, \dots, q_{i-1}, q_i \triangleright q_i, q_{i+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_i, \dots, q_n). \end{aligned}$$

Thus, because of the idempotence of a spindle,  $t_i$  acts as the identity, as desired.

We now turn to the case when  $Q$  is a rack. Since a rack is both a left and right shelf, (a) – (f) remain true. All relations involving inverses follow algebraically from these six. It remains to show that  $s_i$  and  $s_i^{-1}$  and  $t_i$  and  $t_i^{-1}$  are indeed inverses. We have:

$$\begin{aligned} s_i s_i^{-1}(q_1, q_2, \dots, q_n) &= s_i(q_1, q_2, \dots, q_{i-1}, q_{i+1} \triangleleft q_i, q_i, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, q_i, q_i \triangleright (q_{i+1} \triangleleft q_i), q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_i, q_{i+1}, q_{i+2}, \dots, q_n), \end{aligned}$$

where the final equality is due to the right inverse property applied to the  $(i+1)$ st entry. Furthermore,

$$\begin{aligned} s_i^{-1} s_i(q_1, q_2, \dots, q_n) &= s_i^{-1}(q_1, q_2, \dots, q_{i-1}, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, (q_{i+1} \triangleright q_i) \triangleleft q_{i+1}, q_{i+1}, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, q_i, q_{i+1}, \dots, q_n), \end{aligned}$$

where the final equality follows from the left inverse property applied to the  $i$ th entry.

Considering the twists, we have:

$$\begin{aligned} t_i^{-1} t_i(q_1, q_2, \dots, q_n) &= t_i^{-1}(q_1, q_2, \dots, q_{i-1}, q_i \triangleright q_i, q_{i+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, (q_i \triangleright q_i) \triangleleft (q_i \triangleright q_i), q_{i+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_i \triangleright (q_i \triangleleft q_i), \dots, q_n) \\ &= (q_1, q_2, \dots, q_i, \dots, q_n) \end{aligned}$$

where the second to last equality is the mixed distributive law (3) from Lemma 70, and the final equality is due to the left inverse property of a rack. Moreover,

$$\begin{aligned} t_i t_i^{-1}(q_1, q_2, \dots, q_n) &= t_i(q_1, q_2, \dots, q_{i-1}, q_i \triangleleft q_i, q_{i+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, (q_i \triangleleft q_i) \triangleright (q_i \triangleleft q_i), q_{i+1}, \dots, q_n) \\ &= (q_1, q_2, \dots, (q_i \triangleright q_i) \triangleleft q_i, \dots, q_n) \\ &= (q_1, q_2, \dots, q_i, \dots, q_n), \end{aligned}$$

where the second to last equality is the mixed distributive law (4) from Lemma 70, and the final equality is due to the right inverse property of a rack. Thus,  $s_i s_i^{-1} = s_i^{-1} s_i = 1_{Q^n}$  and  $t_i t_i^{-1} = t_i^{-1} t_i = 1_{Q^n}$ , as desired.

Finally, we consider the case when  $Q$  is a quandle. Since a quandle is a spindle, (a) and (b) above still hold. The relations involving inverses follow algebraically from these two. In the previous case we demonstrated that  $s_i$  and  $s_i^{-1}$  are indeed inverses. It remains to show that  $t_i$  acts as the identity. This again follows from the fact that a quandle is a spindle.

Conversely, we show that given a set  $Q$  equipped with operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$ , and defining maps  $s_i, t_i, s_i^{-1}$ , and  $t_i^{-1}$  on  $Q^n$  as above, these maps give an action of

$$\begin{cases} FB_n^+ \\ B_n^+ \\ FB_n \\ B_n \end{cases}$$

on  $Q^n$  if

$$\begin{cases} (Q, \triangleright) \text{ is a quasi-idempotent shelf} \\ (Q, \triangleright) \text{ is a spindle} \\ (Q, \triangleright, \triangleleft) \text{ is a rack} \\ (Q, \triangleright, \triangleleft) \text{ is a quandle.} \end{cases}$$

Consider a set  $Q$  equipped with an operation  $\triangleright: Q \times Q \rightarrow Q$  together with maps  $s_i$  and  $t_i$  defined as above which give an action of  $FB_n^+$ . The relations  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  and  $t_i s_i = s_i t_{i+1}$  imply the distributive and quasi-idempotence laws, so that  $(Q, \triangleright)$  is a quasi-idempotent shelf.

Next, consider a set  $Q$  equipped with an operation  $\triangleright: Q \times Q \rightarrow Q$  and maps  $s_i$  and  $t_i$  defined as above which give an action of  $B_n^+$ . Again, the relation  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  gives the distributive law. Since  $t_i$  acts as the identity in  $B_n^+$ , we have

$$(q_1, q_2, \dots, q_n) = t_i(q_1, q_2, \dots, q_i, \dots, q_n) = (q_1, q_2, \dots, q_{i-1}, q_i \triangleright q_i, q_{i+1}, \dots, q_n)$$

so that  $q_i \triangleright q_i = q_i$ , which is the idempotence condition. Thus,  $(Q, \triangleright)$  is a spindle.

Consider a set  $Q$  equipped with operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$ , together with maps  $s_i, t_i, s_i^{-1}$  and  $t_i^{-1}$  defined as above which give an action of  $FB_n$ . As in the previous two cases, the relation  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  implies the distributive law. Since  $s_i$  and  $s_i^{-1}$  are inverses, we have:

$$\begin{aligned} (q_1, q_2, \dots, q_n) = s_i s_i^{-1}(q_1, q_2, \dots, q_n) &= s_i(q_1, q_2, \dots, q_{i-1}, q_{i+1} \triangleleft q_i, q_i, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, q_i, q_i \triangleright (q_{i+1} \triangleleft q_i), q_{i+2}, \dots, q_n), \end{aligned}$$

and

$$\begin{aligned} (q_1, q_2, \dots, q_n) = s_i^{-1} s_i(q_1, q_2, \dots, q_n) &= s_i^{-1}(q_1, q_2, \dots, q_{i-1}, q_{i+1}, q_{i+1} \triangleright q_i, q_{i+2}, \dots, q_n) \\ &= (q_1, q_2, \dots, q_{i-1}, (q_{i+1} \triangleright q_i) \triangleleft q_{i+1}, q_{i+1}, q_{i+2}, \dots, q_n), \end{aligned}$$

so that  $q_i \triangleright (q_{i+1} \triangleleft q_i) = q_{i+1}$  and  $(q_{i+1} \triangleright q_i) \triangleleft q_{i+1} = q_i$ , which are the two inverse properties. Thus,  $(Q, \triangleright, \triangleleft)$  is a rack.

Finally, consider a set  $Q$  equipped with operations  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$ , together with maps  $s_i, t_i, s_i^{-1}$  and  $t_i^{-1}$  defined as above which give an action of  $B_n$ . The distributive law follows from the relation  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ , and the two inverse properties follow from the fact that  $s_i$  and  $s_i^{-1}$  are inverses. As in the spindle case, the idempotence law is a result of the fact that  $t_i$  acts as the identity. Thus,  $(Q, \triangleright, \triangleleft)$  is a quandle.  $\square$

The moral of this theorem is that the relations in these four braid and framed braid groups and monoids *force* the shelf, rack, spindle, and quandle laws to hold! Therefore, if we desired, we could actually *define* a shelf, rack, spindle, or quandle to be a set equipped with binary operations,  $(Q, \triangleright, \triangleleft)$ , such that if we defined maps on  $Q^n$  by the above formulas, we would obtain an action of the appropriate braid or framed braid group or monoid. Doing so would enable us to easily categorify these concepts, since the categorified versions of the braid and framed braid groups and monoids are known.

Now that we have exhibited the connection between the categories of shelves, racks, spindles and quandles, we will use them to describe a new means by which we can obtain the Lie algebra of a Lie group. Specifically, we will show that we can think of our Lie group as a spindle in  $\text{Diff}_*$ , just as any group gives a spindle. Therefore, we continue in the next section by internalizing the concepts of shelf, rack, spindle and quandle.

### 3.1.4 Internalization

Our goal is to present a conceptual explanation of the passage from a Lie group to its Lie algebra using the language of spindles. Seeing as how the bracket arises from differentiating conjugation, and spindles possess essential properties of conjugation, we desire a way to treat our Lie groups as though they were spindles. This idea should not be so surprising since we saw in Section 3.1.2 that the self-distributive law of a shelf and the Jacobi identity of a Lie algebra are each equivalent to the Yang–Baxter equation! As mentioned previously, however, in addition to the self-distributive law of a shelf, we need the idempotence law of a spindle to obtain the antisymmetry of the bracket.

Using the language of internalization, as was done in Section 2.2 of Chapter 2 to define the notion of a 2-vector space, a Lie group can be thought of as a ‘group in  $\text{Diff}_*$ ’. Then, we can internalize the relationship between groups and spindles to show that a Lie group is also a ‘spindle in  $\text{Diff}_*$ ’. Though we will only use the internalized version of a spindle, we introduce the internalized versions of all of the concepts from Section 3.1.1 for completeness. In the following, we consider a category  $K$  with finite products.

**Definition 75.** *A left shelf in  $K$  consists of:*

- *an object  $Q \in K$ ,*

*equipped with:*

- *a left conjugation morphism  $\triangleright: Q \times Q \rightarrow Q$ ,*

*such that the following diagram commutes, expressing the usual left distributive law:*

- the left distributive law, (i'):

$$\begin{array}{ccccc}
& & Q \times Q \times Q & & \\
& \swarrow \Delta \times 1 \times 1 & & \searrow 1 \times \triangleright & \\
Q \times Q \times Q \times Q & & & & Q \times Q \\
\downarrow 1 \times S \times 1 & & & & \downarrow \triangleright \\
Q \times Q \times Q \times Q & & & & Q \\
& \searrow \triangleright \times 1 \times 1 & & \nearrow \triangleright & \\
& Q \times Q \times Q & \xrightarrow{1 \times \triangleright} & Q \times Q &
\end{array}$$

where  $\Delta: Q \rightarrow Q \times Q$  is the diagonal morphism in  $K$  and  $S: Q \times Q \rightarrow Q \times Q$  is a morphism which switches its inputs.

**Definition 76.** A right shelf in  $K$  consists of:

- an object  $Q \in K$ ,

equipped with:

- a **right conjugation** morphism  $\triangleleft: Q \times Q \rightarrow Q$ ,

such that the following diagram commutes, expressing the usual right distributive law:

- the right distributive law, (ii'):

$$\begin{array}{ccccc}
& & Q \times Q \times Q & & \\
& \swarrow 1 \times 1 \times \Delta & & \searrow \triangleleft \times 1 & \\
Q \times Q \times Q \times Q & & & & Q \times Q \\
\downarrow 1 \times S \times 1 & & & & \downarrow \triangleleft \\
Q \times Q \times Q \times Q & & & & Q \\
& \searrow 1 \times 1 \times \triangleleft & & \nearrow \triangleleft & \\
& Q \times Q \times Q & \xrightarrow{\triangleleft \times 1} & Q \times Q &
\end{array}$$

where  $\Delta: Q \rightarrow Q \times Q$  is the diagonal morphism in  $K$  and  $S: Q \times Q \rightarrow Q \times Q$  is a morphism which switches its inputs.

**Definition 77.** A rack in  $K$  consists of:

- an object  $Q \in K$ ,

equipped with:

- a **left conjugation** morphism  $\triangleright: Q \times Q \rightarrow Q$ ,

- a **right conjugation morphism**  $\triangleleft: Q \times Q \rightarrow Q$ ,

such that  $(Q, \triangleright)$  is a left shelf in  $K$ ,  $(Q, \triangleleft)$  is a right shelf in  $K$ , and the following diagrams commute, expressing the usual left and right inverse laws:

- the **left inverse law**, (iii'):

$$\begin{array}{ccccc}
 Q \times Q & \xrightarrow{\Delta \times 1} & Q \times Q \times Q & \xrightarrow{1 \times S} & Q \times Q \times Q \\
 & \searrow \pi_2 & & & \downarrow \triangleright \times 1 \\
 & & & & Q \times Q \\
 & & & & \downarrow \triangleleft \\
 & & & & Q
 \end{array}$$

- the **right inverse law**, (iv'):

$$\begin{array}{ccccc}
 Q \times Q & \xrightarrow{\Delta \times 1} & Q \times Q \times Q & \xrightarrow{1 \times S} & Q \times Q \times Q \\
 & \searrow \pi_2 & & & \downarrow 1 \times \triangleleft \\
 & & & & Q \times Q \\
 & & & & \downarrow \triangleright \\
 & & & & Q
 \end{array}$$

where  $\Delta: Q \rightarrow Q \times Q$  is the diagonal morphism in  $K$  and  $S: Q \times Q \rightarrow Q \times Q$  is a morphism which switches its inputs.

**Definition 78.** A left spindle in  $K$  consists of:

- an object  $Q \in K$ ,

equipped with:

- a **left conjugation morphism**  $\triangleright: Q \times Q \rightarrow Q$ ,

such that  $(Q, \triangleright)$  is a left shelf in  $K$  and the following diagram commutes, expressing the usual left idempotence law:

- the **left idempotence law**, (v'):

$$\begin{array}{ccc}
 & Q \times Q & \\
 \Delta \nearrow & & \downarrow \triangleright \\
 Q & \xrightarrow{1} & Q
 \end{array}$$

where  $\Delta: Q \rightarrow Q \times Q$  is the diagonal morphism in  $K$ .

**Definition 79.** A quandle in  $K$  consists of:

- an object  $Q \in K$ ,

equipped with:

- a **left conjugation morphism**  $\triangleright: Q \times Q \rightarrow Q$ ,
- a **right conjugation morphism**  $\triangleleft: Q \times Q \rightarrow Q$ ,

such that  $(Q, \triangleright, \triangleleft)$  is a rack in  $K$  and the following diagrams commute, expressing the usual left and right idempotence laws:

- the **left idempotence law**,  $(v')$ :

$$\begin{array}{ccc} & & Q \times Q \\ & \nearrow \Delta & \downarrow \triangleright \\ Q & \xrightarrow{1} & Q \end{array}$$

- the **right idempotence law**,  $(vi')$ :

$$\begin{array}{ccc} & & Q \times Q \\ & \nearrow \Delta & \downarrow \triangleleft \\ Q & \xrightarrow{1} & Q \end{array}$$

where  $\Delta: Q \rightarrow Q \times Q$  is the diagonal morphism in  $K$ .

In the special case where  $K = \text{Set}$ , these concepts in  $K$  reduce to the ordinary notions of shelf, rack, spindle and quandle. Therefore, we should expect that we can internalize the close relationship between quandles and groups, and show that a group in  $K$ , a category with finite products, gives a quandle in  $K$ . We first recall the definition of an internalized group:

**Definition 80.** A group in  $K$  consists of:

- an object  $G \in K$ ,

equipped with:

- a **multiplication morphism**  $m: G \times G \rightarrow G$
- an **identity morphism**  $\text{id}: I \rightarrow G$ , where  $I$  is the terminal object in  $K$
- an **inverse morphism**  $\text{inv}: G \rightarrow G$ ,

such that the following diagrams commute, expressing the usual group laws:



- the associative law:

$$\begin{array}{ccc}
 & G \times G \times G & \\
 m \times 1 \swarrow & & \searrow 1 \times m \\
 G \times G & & G \times G \\
 m \searrow & & \swarrow m \\
 & G &
 \end{array}$$

- the right and left unit laws:

$$\begin{array}{ccccc}
 I \times G & \xrightarrow{\text{id} \times 1} & G \times G & \xleftarrow{1 \times \text{id}} & G \times I \\
 & \searrow & \downarrow m & \swarrow & \\
 & & G & &
 \end{array}$$

- the right and left inverse laws:

$$\begin{array}{ccc}
 G \times G & \xrightarrow{1 \times \text{inv}} & G \times G \\
 \Delta \nearrow & & \searrow m \\
 G & & G \\
 \searrow & \nearrow \text{id} & \\
 & I &
 \end{array}
 \quad
 \begin{array}{ccc}
 G \times G & \xrightarrow{\text{inv} \times 1} & G \times G \\
 \Delta \nearrow & & \searrow m \\
 G & & G \\
 \searrow & \nearrow \text{id} & \\
 & I &
 \end{array}$$

where  $\Delta: G \rightarrow G \times G$  is the diagonal morphism in  $K$ .

Indeed, for any category  $K$  with finite products, there is a category  $K\text{Grp}$  consisting of groups in  $K$  and homomorphisms between these, where a **homomorphism**  $f: G \rightarrow G'$  is a morphism in  $K$  that preserves multiplication, meaning that this diagram commutes:

$$\begin{array}{ccc}
 G \times G & \xrightarrow{m} & G \\
 f \times f \downarrow & & \downarrow f \\
 G' \times G' & \xrightarrow{m'} & G'
 \end{array}$$

Given a group in  $K$ , say  $G$ , we can construct a quandle in  $K$  by defining the left and right conjugation morphisms in terms of conjugation in  $G$ , just as we did when  $K = \text{Set}$ . That is, we define left conjugation  $\triangleright: G \times G \rightarrow G$  by the composite:

$$(x, y) \xrightarrow{\Delta \times 1} (x, x, y) \xrightarrow{1 \times S} (x, y, x) \xrightarrow{1 \times 1 \times \text{inv}} (x, y, x^{-1}) \xrightarrow{1 \times m} (x, yx^{-1}) \xrightarrow{m} xyx^{-1}$$

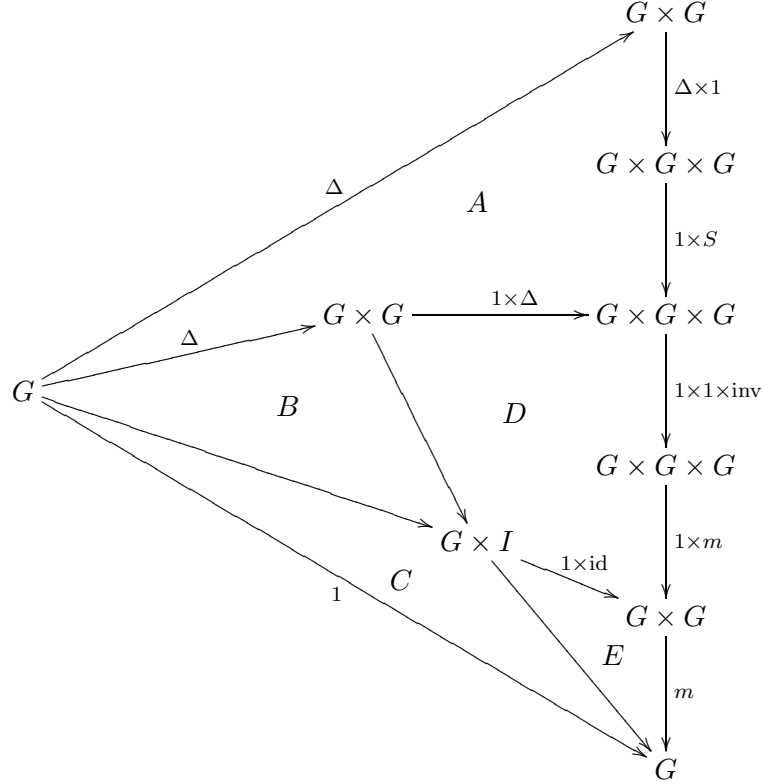
while right conjugation  $\triangleleft: G \times G \rightarrow G$  is defined by the composite:

$$(x, y) \xrightarrow{1 \times \Delta} (x, y, y) \xrightarrow{S \times 1} (y, x, y) \xrightarrow{\text{inv} \times 1 \times 1} (y^{-1}, x, y) \xrightarrow{m \times 1} (y^{-1}x, y) \xrightarrow{m} y^{-1}xy$$

We show that these operations satisfy the quandle axioms by simply internalizing the calculations which showed that left and right conjugation in groups satisfy the quandle laws. For example, the left idempotence law for left conjugation holds since

$$x \triangleright x = xxx^{-1} = x1 = x,$$

so that we expect the internalized version to use the internalized right inverse and left unit laws. The left idempotence law for a quandle in  $K$  follows from this diagram:



Regions  $A$ ,  $B$ , and  $C$  commute trivially. Region  $D$  is the right inverse law multiplied on the left by 1, and region  $E$  is the left unit law.

The remaining internalized quandle axioms follow from similar sorts of diagrams. Thus, we obtain quandles in  $K$  from groups in  $K$  just as we get quandles from groups. We continue by showing that there is an analogue of Proposition 68 for rack objects. As in Section 3.1.2, we begin by illustrating the relationship between the left shelf axiom and the Yang–Baxter equation:

**Lemma 81.** *The braiding  $B: Q \times Q \rightarrow Q \times Q$  defined as  $B = S(\Delta \times 1)(1 \times \triangleright)$  satisfies the Yang–Baxter equation if and only if  $(Q, \triangleright)$  is a left shelf object.*

**Proof.** The proof of this statement amounts to a rather large, though mostly trivial, diagram that is nothing more than an internalization of the proof of Lemma 61.  $\square$

Moreover, internalizing the proofs of Lemma 62, Corollary 66, and Lemma 67 results in the following:

**Lemma 82.** *The inverse braiding,  $B^{-1} = S(1 \times \Delta)(\triangleleft \times 1)$ , satisfies the Yang–Baxter equation if and only if  $(Q, \triangleleft)$  is a right shelf object.*

**Lemma 83.** *Let  $Q$  be an object of a category  $K$  with finite products equipped with two morphisms  $\triangleright: Q \times Q \rightarrow Q$  and  $\triangleleft: Q \times Q \rightarrow Q$  such that  $(Q, \triangleright)$  is a left shelf in  $K$  and  $(Q, \triangleleft)$  is a right shelf in  $K$ . Then, the braidings  $B$  and  $B^{-1}$  are inverses if and only if  $(Q, \triangleright, \triangleleft)$  is a rack in  $K$ .*

**Lemma 84.** *If  $B$  is invertible and satisfies the Yang–Baxter equation, then  $B^{-1}$  does as well.*

Thus, a shelf in  $K$  together with an additional morphism which satisfies the two internalized inverse properties,  $(iii')$  and  $(iv')$ , is a rack in  $K$ . This statement follows from the previous lemmas as in the case of racks.

**Proposition 85.** *A left shelf  $(Q, \triangleright)$  in  $K$  together with a morphism  $\triangleleft: Q \times Q \rightarrow Q$  satisfying the internalized inverse properties  $(iii')$  and  $(iv')$  also satisfies the internalized right distributive law,  $(ii')$ . That is,  $(Q, \triangleright, \triangleleft)$  is a rack in  $K$ .*

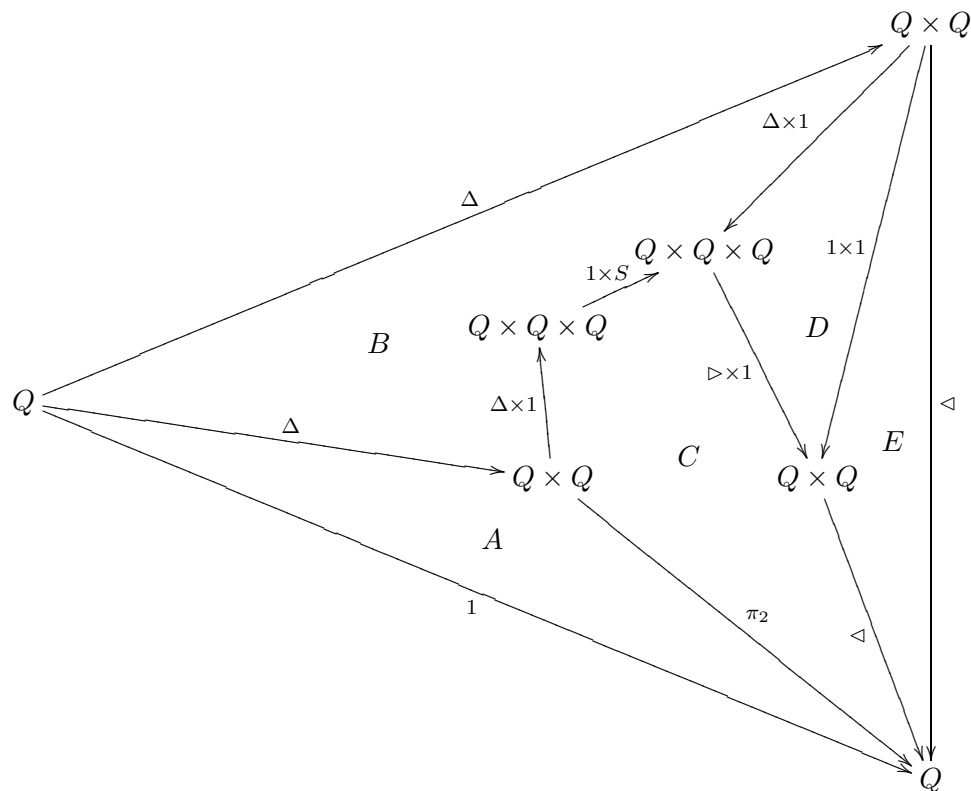
Just as we demonstrated that we could obtain one of the idempotent laws from the other for quandles, the corresponding result holds for quandles in  $K$ . Now, however, instead of equations, we have commutative diagrams.

**Proposition 86.** *In the definition of a quandle in  $K$ , the left idempotence law,  $(v')$ , is satisfied if and only if the right idempotence law,  $(vi')$ , is satisfied.*

**Proof.** We prove that the left idempotence law implies the right. That is, we will show that diagram  $(vi')$  commutes using the two inverse properties,  $(iii')$  and  $(v')$ . The other implication is proved analogously.

In the figure below, regions  $A$ ,  $B$ , and  $E$  clearly commute. Region  $C$  commutes because it is the internalized left inverse property, axiom  $(iii')$ , and region  $D$  results when multiplying the internalized left idempotence property, axiom  $(v')$ , on the right by 1 and

therefore commutes:



Thus, since each piece of the diagram commutes, the outer edges do as well, which gives  $(vi')$ .  $\square$

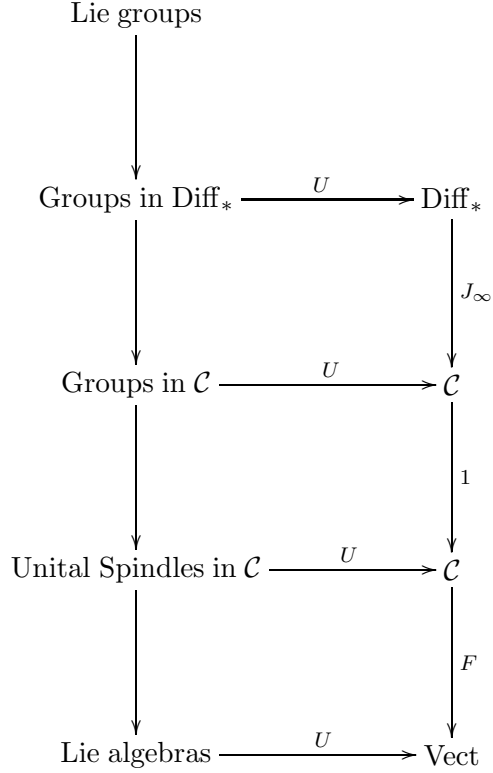
In Theorem 74 from Section 3.1.3, we demonstrated how shelves, racks, spindles, and quandles give actions of the various braid and framed braid groups and monoids in the category of **Set**. In fact, this result generalizes in an obvious way to shelves, racks, spindles, and quandles in a category with finite products. The generalized proof is just like the proof given for sets where now everything is internalized, just like the proof of the previous proposition.

Now that we have the ability to view our Lie groups as spindles in  $\text{Diff}_*$ , we continue by developing the additional necessary machinery we need to obtain the corresponding Lie algebras.

## 3.2 From Lie Groups to Lie Algebras

Recall that our goal is to describe a novel, conceptual explanation of the passage from a Lie group to its Lie algebra using the language of spindles. More specifically, we will use the internalized concepts of the previous section to think of a Lie group as a group, and therefore, spindle in  $\text{Diff}_*$ , and then show how to use this spindle in  $\text{Diff}_*$  to obtain the Lie algebra of the given Lie group. Recall from the introduction to this chapter that our

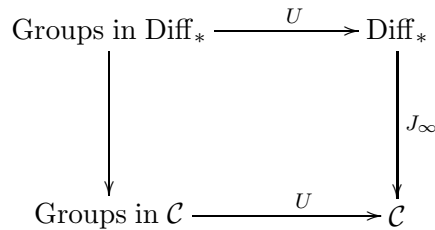
desired process takes the following diagrammatic form:



The first tasks consist of defining the ‘cojets’ functor  $J_\infty$  and category of ‘special coalgebras’,  $\mathcal{C}$ .

### 3.2.1 Cojets

In the next two sections, we focus on this aspect of our diagram:



by defining the ‘cojets’ functor,  $J_\infty$ , and the category  $\mathcal{C}$  of ‘special coalgebras’.

Since we are interested in obtaining the Lie algebra of a Lie group, and since the bracket in the Lie algebra arises from differentiating conjugation in the Lie group *twice*, we need a way to keep track of higher-order derivatives. Roughly speaking, a ‘ $k$ -jet’ of a smooth function  $f$  is a gadget which keeps track of the value of the  $f$  at a point together with all its derivatives up to the  $k$ th order. In other words, it is simply a way of describing the Taylor expansion of  $f$  at a given point up to order  $k$ .

Let  $M$  be a smooth manifold and let  $C^\infty(M)$  be the algebra of smooth real-valued functions on  $M$ . Suppose  $p$  is a point of  $M$  and let  $f \in C^\infty(M)$ . Recall that the first partial derivatives of  $f$  define a smooth map  $df: TM \rightarrow \mathbb{R}$ .

**Definition 87.**  $f$  **vanishes to zeroth order at  $p$**  if  $f(p) = 0$

**Definition 88.**  $f$  **vanishes to first order at  $p$**  if it vanishes to zeroth order at  $p$  and  $df: TM \rightarrow \mathbb{R}$  vanishes to zeroth order at every point of  $T_pM$ .

**Definition 89.**  $f$  **vanishes to  $k$ th order at  $p$**  if it vanishes to  $(k-1)$ st order at  $p$  and  $df: TM \rightarrow \mathbb{R}$  vanishes to  $(k-1)$ st order at every point of  $T_pM$ .

Saying that  $f$  vanishes to  $k$ th order simply means that  $f$  vanishes along with all its partial derivatives up to the  $k$ th order at  $p$ ; the fancier definition given above simply shows that this concept is independent of any choice of coordinates.

**Definition 90.** Let  $I^k(M, p)$  be the set of all smooth functions  $f: M \rightarrow \mathbb{R}$  that vanish to  $k$ th order at  $p$ .

The product rule implies that  $I^k(M, p)$  is an ideal in  $C^\infty(M)$ . This lets us define a finite-dimensional quotient algebra, which we call the algebra of ‘ $k$ -jets’:

**Definition 91.** Let  $J^k(M, p)$  denote the algebra  $C^\infty(M)/I^k(M, p)$ . We call an element of  $J^k(M, p)$  a  **$k$ -jet at  $p$** . Given a function  $f \in C^\infty(M)$ , we denote its equivalence class in  $J^k(M, p)$  by  $[f]_k$ , and call this the  **$k$ -jet of  $f$  at  $p$** .

Thus,  $[f]_k$  consists of all functions whose derivatives up to order  $k$  agree at  $p$ . Moreover, the Taylor expansions of degree  $k$  at  $p$  of functions in  $[f]_k$  agree.

We now define a contravariant functor  $J^k: \text{Diff}_* \rightarrow \text{Vect}$  which sends  $(M, p) \in \text{Diff}_*$  to  $J^k(M, p)$ , the algebra of  $k$ -jets of real-valued functions on  $M$  at the point  $p$ . To define  $J^k$  on morphisms, we consider a smooth map  $f: (M, p) \rightarrow (N, q)$ , and define  $J^k(f): J^k(N, q) \rightarrow J^k(M, p)$  by setting  $J^k(f)[\phi]_k = [f\phi]_k$ , where  $[\phi]_k$  is the  $k$ -jet at some point  $q$  of the function  $\phi: N \rightarrow \mathbb{R}$ . This definition is independent of the choice of representative of the equivalence class  $[\phi]_k$  since, by the chain rule, the first  $k$  derivatives of  $f\phi$  are determined by the first  $k$  derivatives of  $\phi$  so long as we know  $f$ . The contravariance and functoriality of  $J^k$  follow from routine computations.

We have defined  $J^k(M, p)$  in a completely coordinate-independent way, making it seem somewhat mysterious. We know that  $J^k(M, p)$  contains all the information about the derivatives of real valued functions on  $M$  at  $p$  up to order  $k$ . Therefore, it should not come as a surprise that the space of  $k$ -jets is isomorphic to the space of Taylor polynomials of degree  $k$ , since  $k$ th order Taylor expansions keep track of all partial derivatives up to order  $k$ . Furthermore, given a real vector space  $V$ , the symmetric algebra,  $SV$ , is isomorphic to the polynomial algebra in  $n$  variables, so we can exhibit a relationship between the space of  $k$ -jets and the sum of the spaces of symmetric tensors of degree less than or equal to  $k$ . This association will allow us to understand what the space of  $k$ -jets ‘looks like’.

**Lemma 92.** *The space of  $k$ -jets of  $f$  at  $p$ ,  $J^k(M, p)$ , is isomorphic as an algebra to  $\bigoplus_{i=0}^k S^i(T_p^* M)$ .*

**Proof.** If  $\dim(M) = n$ , then  $S^k(T_p^* M)$  is isomorphic to the algebra of polynomials in  $n$  variables of degrees less than or equal to  $k$ . Choose coordinates  $x_1, x_2, \dots, x_n$  on a neighborhood of  $p \in M$ . To simplify notation, we will use  $D_i f$  to denote  $\frac{\partial f}{\partial x_i}$ . Define a map

$$\phi: J^k(M, p) \rightarrow \bigoplus_{i=0}^k S^i(T_p^* M)$$

which sends a  $k$ -jet,  $[f]_k$ , to its Taylor polynomial of degree  $k$ :

$$\phi([f]_k) = \sum \frac{(D_1^{i_1} \cdots D_n^{i_n} f)(p)}{i_1! \cdots i_n!} x_1^{i_1} \cdots x_n^{i_n},$$

where this summation extends over all ordered  $n$ -tuples  $(i_1, \dots, i_n)$  such that each  $i_j$  is a nonnegative integer and  $i_1 + \cdots + i_n \leq k$ . This map depends on our choice of coordinates and becomes well-defined once we have made our choice. The map  $\phi$  is injective since all functions within the same equivalence class have the same derivatives, and therefore Taylor expansions, up to order  $k$  at  $p$ .

To show that  $\phi$  is surjective, we first note that any polynomial of degree  $k$  is the Taylor expansion of degree  $k$  of some compactly supported smooth function on  $\mathbb{R}^n$ . Given a polynomial of degree  $k$ , we can multiply it by a smooth ‘bump’ function that is defined to be 1 in a neighborhood of the origin and vanishes outside of some large ball. By construction, the product of our polynomial with this bump function has the same derivatives at the origin as our original polynomial. Moreover, given a manifold  $M$  of dimension  $n$  and coordinate chart  $U \ni p$ , any polynomial  $q(x)$  of degree  $k$  is the Taylor expansion of degree  $k$  at  $p$  of some smooth function on  $M$ . To see this, we can use our first observation to construct a compactly supported smooth function  $f$  on  $U$  such that  $q(x)$  is the  $k$ th degree Taylor polynomial of  $f$ . We can then extend  $f$  to a smooth function  $\tilde{f}$  on all of  $M$  by defining  $\tilde{f}(x) = 0$  for all  $x \in M \setminus U$ . Thus,  $q(x)$  is the Taylor expansion of degree  $k$  of  $\tilde{f}$  on  $M$ , so that  $\phi$  is surjective.

Furthermore, these spaces are isomorphic as algebras. The multiplication on  $J^k(M, p)$  is defined as  $[f]_k \cdot [g]_k := [fg]_k$ , while the multiplication on  $\bigoplus_{i=0}^k S^i(T_p^* M)$  amounts to first multiplying two polynomials,  $q \cdot r$ , and then forming the Taylor expansion of degree  $k$  of the product,  $T_k(qr)$ , at  $p$ . That is, we multiply the polynomials  $q$  and  $r$  and then throw out the terms having degree higher than  $k$ . Thus, the isomorphism  $\phi$  is a morphism of algebras since first multiplying two  $k$ -jets and then forming the Taylor polynomial of degree  $k$  is the same as first forming the Taylor polynomials of the two  $k$ -jets and then generating the  $k$ th degree Taylor polynomial of their product.  $\square$

Recall that our goal is to construct a product preserving functor  $J_\infty: \text{Diff}_* \rightarrow \mathcal{C}$ , where  $\mathcal{C}$  will be the category of ‘special coalgebras’, that sends a pointed manifold  $(M, p)$

to the coalgebra of ‘ $k$ -cojets at  $p$ ’. In order to preserve products, this functor must be covariant. Before we continue toward our goal, we recall the definition of a coalgebra, which is the dual notion of an algebra.

**Definition 93.** A **coalgebra** is a vector space  $C$  together with a **comultiplication**  $\Delta: C \rightarrow C \otimes C$  which is bilinear and coassociative. A **coalgebra with counit** is a coalgebra with a **counit**  $\epsilon: C \rightarrow k$  such that

$$\begin{array}{ccccc} k \times C & \xrightarrow{\epsilon \times 1} & C \times C & \xleftarrow{1 \times \epsilon} & C \times k \\ & \searrow & \uparrow \Delta & \swarrow & \\ & & C & & \end{array}$$

commutes. A coalgebra is **cocommutative** if  $\Delta(c) = S(\Delta(c))$  for  $c \in C$  where  $S: C \otimes C \rightarrow C \otimes C$  is the map that switches its inputs.

**Definition 94.** Given two coalgebras  $C$  and  $D$ ,  $f: C \rightarrow D$  is a **coalgebra homomorphism** if  $f$  is linear and the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes C \\ f \downarrow & & \downarrow f \otimes f \\ D & \xrightarrow{\Delta_D} & D \otimes D \end{array}$$
  

$$\begin{array}{ccc} C & \xrightarrow{\epsilon_C} & k \\ f \searrow & & \uparrow \epsilon_D \\ & & D \end{array}$$

We remark that the dual of an algebra is a coalgebra so long as the algebra is finite-dimensional. To see this, consider an algebra  $A$  with multiplication  $m: A \otimes A \rightarrow A$ . Then,  $A^*$  becomes a coalgebra with comultiplication  $\Delta: A^* \rightarrow A^* \otimes A^*$  defined as the composite:

$$\begin{array}{ccc} A^* & \xrightarrow{m^*} & (A \otimes A)^* \\ & \searrow \Delta & \downarrow \wr \\ & & A^* \otimes A^* \end{array}$$

where the isomorphism from  $(A \otimes A)^*$  to  $A^* \otimes A^*$  exists only when  $A$  is finite-dimensional.

Therefore, seeing as how we have a finite-dimensional *algebra*,  $J^k(M, p)$ , and *contravariant* functor,  $J^k$ , it is natural to consider the duals of these as a first step toward achieving our goal. Therefore, we consider

$$J_k(M, p) = J^k(M, p)^*,$$



which is the space of ***k*-cojets at *p***. Since  $J_k(M, p)$  is the dual of a finite-dimensional algebra, it is a coalgebra. Furthermore, we define the functor  $J_k: \text{Diff}_* \rightarrow \text{Vect}$  which sends any object  $(M, p) \in \text{Diff}_*$  to  $J_k(M, p)$  and any smooth map  $f: (M, p) \rightarrow (N, q)$  to  $J_k(f): J_k(M, p) \rightarrow J_k(N, q)$ , which is the adjoint of the linear map  $J^k(f): J^k(N, q) \rightarrow J^k(M, p)$ . The functoriality and covariance of  $J_k$  hold since it is the adjoint of a contravariant functor.

Lemma 92 established an algebra isomorphism from the space of *k*-jets to the space  $\bigoplus_{i=0}^k S^i(T_p^* M)$ . The dual statement gives us an isomorphism of coalgebras:

**Lemma 95.** *The space of *k*-cojets of *f* at *p*,  $J_k(M, p)$  is isomorphic as a coalgebra to  $\bigoplus_{i=0}^k S^i(T_p M)$ .*

**Proof.**

$$\begin{aligned}
 J_k(M, p) = J^k(M, p)^* &\cong \left( \bigoplus_{i=0}^k S^i(T_p^* M) \right)^* \\
 &\cong \bigoplus_{i=0}^k (S^i(T_p^* M))^* \\
 &\cong \bigoplus_{i=0}^k S^i(T_p^{**} M) \\
 &\cong \bigoplus_{i=0}^k S^i(T_p M)
 \end{aligned}$$

□

Thus far, we have achieved precisely what we desired: a covariant functor  $J_k$  that sends any pointed smooth manifold  $(M, p)$  to the coalgebra of *k*-cojets of real-valued functions on *M* at the point *p*. Unfortunately, however, this functor does not preserve products since

$$\begin{aligned}
 J_k(M \times N, (p, q)) &\cong \bigoplus_{i=0}^k S^i(T_{(p,q)}(M \times N)) \\
 &\cong \bigoplus_{i=0}^k S^i(T_p M \oplus T_q(N)) \\
 &\cong \bigoplus_{i=0}^k \bigoplus_{j=0}^i S^j(T_p M) \otimes S^{i-j}(T_q N) \\
 &\not\cong \bigoplus_{i=0}^k S^i(T_p M) \otimes \bigoplus_{j=0}^k S^j(T_q N) \\
 &\cong J_k(M, p) \otimes J_k(N, q)
 \end{aligned}$$

Failing to preserve products implies that  $J_k$  will not preserve groups, and hence spindles, which is crucial. In order to overcome this shortcoming, we consider a space that more closely resembles the symmetric algebra, since it preserves products. Thus, we turn to the task of constructing a product-preserving functor that will send a pointed smooth manifold  $(M, p)$  to the coalgebra of ‘cojets’: the union of the coalgebras of  $k$ -cojets.

For each  $k$ , we have a surjective map from  $J^{k+1}(M, p)$  to  $J^k(M, p)$  which simply forgets the  $(k + 1)$ st partial derivatives. More precisely, this map arises as a result of the fact that we mod out by more when we form the quotient algebra  $J^k(M, p)$  than when we form the algebra  $J^{k+1}(M, p)$ . Thus we obtain a diagram:

$$\dots \rightarrow J^3(M, p) \rightarrow J^2(M, p) \rightarrow J^1(M, p) \rightarrow J^0(M, p) = \mathbb{R}$$

where each arrow is surjective.

Taking the dual, or adjoint, of this diagram produces a corresponding inclusion of coalgebras of  $k$ -cojets. Forming the dual amounts to turning the arrows around and taking the duals of the algebras of  $k$ -jets. We obtain:

$$\mathbb{R} = J_0(M, p) \rightarrow J_1(M, p) \rightarrow J_2(M, p) \rightarrow J_3(M, p) \rightarrow \dots$$

where now each arrow is injective. Thus, we can think of  $J_k(M, p)$  as sitting inside of  $J_{k+1}(M, p)$  and therefore we may form the category-theoretic colimit, or union, of these spaces

$$J_\infty(M, p) := \bigcup_{k \geq 0} J_k(M, p),$$

which we call the **space of cojets at  $p$** .

We remark that we could have formed the corresponding space of jets,  $J^\infty(M, p)$ , by taking the limit of the diagram of  $k$ -jets above. However, unlike in the case of  $k$ -jets and  $k$ -cojets, the dual of the space of jets is not isomorphic to the space of cojets. That is,  $J_\infty(M, p) \neq J^\infty(M, p)^*$ .

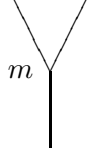
The space  $J_\infty(M, p)$  of cojets clearly is a vector space, as it is the union of  $J_k(M, p)$  which are vector spaces. In fact, the categorically inclined reader will notice that  $J_\infty(M, p)$  is just the colimit of the coalgebras  $J_k(M, p)$ ! That is,  $J_\infty(M, p)$  has a much richer structure: that of a cocommutative coalgebra. In the next section, we describe the coalgebra structures of  $J_\infty(M, p)$  and the symmetric algebra  $S(T_p M)$  and show that they are compatible.

### 3.2.2 Special Coalgebras

In this section, we describe the coalgebra structures of the space of cojets and the symmetric algebra, and then provide a definition of the category  $\mathcal{C}$  of ‘special coalgebras’, which is the target category of our functor  $J_\infty$ .

Since we have already recalled the algebraic definition of a coalgebra, we now offer a diagrammatic description of both algebras and coalgebras. Since multiplication is a process that takes two things and fuses them together into one new thing of the same sort,

we can represent this process diagrammatically as:



Then, since comultiplication is the dual process, we ‘cothink’ and turn everything upside down! Thus, comultiplication takes the form:



In terms of these diagrams, associativity and coassociativity are expressed as:



while commutativity and cocommutativity can be depicted as:



With these diagrams, it becomes a simple exercise to show that the tensor product of two commutative algebras is again a commutative algebra, and the tensor product of two cocommutative coalgebras is again a cocommutative coalgebra.

We have shown that the space of cojets is isomorphic as a vector space to the symmetric algebra, but, as already noted, both  $J_\infty(M, p)$  and  $S(T_p M)$  have much richer structures! They are both cocommutative coalgebras, with a compatible comultiplication. The symmetric algebra becomes a coalgebra with comultiplication  $\Delta: S(T_p M) \rightarrow S(T_p M) \otimes S(T_p M)$  defined as  $\Delta(v) = v \otimes 1 + 1 \otimes v$  for  $v \in T_p M$ . Notice that this determines  $\Delta$  on all of  $S(T_p M)$  since  $\Delta$  is a morphism of algebras and elements  $v \in T_p M$  generate all of  $S(T_p M)$  as an algebra. Thus, knowing what  $\Delta$  does to elements in  $T_p M$  allows us to determine  $\Delta$  on all of  $S(T_p M)$ . For example, we must have  $\Delta(1) = 1 \otimes 1$  since  $\Delta$  is an algebra homomorphism and  $1 \otimes 1$  is the multiplicative identity in the algebra  $S(T_p M) \otimes S(T_p M)$ . We must also have  $\Delta(vw) = \Delta(v)\Delta(w)$  for  $v, w \in T_p M$ . Similarly, the counit  $\epsilon: S(T_p M) \rightarrow k$  is defined by  $\epsilon(v) = 0$  for  $v \in T_p M$ , and computations show that this determines  $\epsilon$  on all of  $S(T_p M)$ . Moreover,  $S(T_p M)$  is a cocommutative coalgebra, since

$$\Delta(v) = v \otimes 1 + 1 \otimes v = 1 \otimes v + v \otimes 1 = S(v \otimes 1 + 1 \otimes v) = S(\Delta(v)),$$

where  $S: T_p M \otimes T_p M \rightarrow T_p M \otimes T_p M$  is the map that switches its inputs. Of course, none of this is particular to the tangent space,  $T_p M$ , and holds for an arbitrary vector space  $V$ .

Furthermore,  $J_\infty(M, p)$  becomes a coalgebra, as it is the union over  $k$  of the coalgebras  $J_k(M, p)$ , with comultiplication and counit defined in terms of these operations on the appropriate  $J_k(M, p)$ . Given  $[f] \in J_\infty(M, p) = \bigcup_k J_k(M, p)$ ,  $[f] \in J_k(M, p)$  for some  $k$ , so we define  $\Delta[f] = \Delta_k[f]$  and  $\epsilon[f] = \epsilon_k[f]$ . Recall that  $\Delta_k$  and  $\epsilon_k$  are defined as the adjoints of the multiplication and unit maps on  $J^k(M, p)$  since  $J_k(M, p) = J^k(M, p)^*$ . Like  $S(T_p M)$ ,  $J_\infty(M, p)$  is a cocommutative coalgebra, which can be seen from a computation similar to the one given above for  $S(T_p M)$ .

Thus, any of the coordinate-dependent isomorphisms in Lemma 95 give rise to coalgebra isomorphisms:

**Lemma 96.** *The space of cojets  $J_\infty(M, p)$  is isomorphic as a cocommutative coalgebra to the symmetric algebra  $S(T_p M)$ .*

**Proof.**

$$J_\infty(M, p) = \bigcup_k J_k(M, p) \cong \bigcup_k \bigoplus_{i=0}^k S^i(T_p M) \cong \bigoplus_{i=0}^{\infty} S^i(T_p M) = S(T_p M) \quad \square$$

In fact, we remark that both  $SV$  and  $J_\infty(M, p)$  are *bialgebras*, meaning that they are algebras and coalgebras in a compatible way. That is, the comultiplication and counit are *algebra* homomorphisms.

Now that we know the coalgebra of cojets is isomorphic to the symmetric algebra, and since the symmetric algebra functor is product-preserving, we can use this to construct the target category,  $\mathcal{C}$ , of our functor  $J_\infty$ .

**Definition 97.** *We define  $\mathcal{C}$ , the category of **special coalgebras**, to be the category whose objects are triples  $(C, V, \alpha)$  where  $C$  is a cocommutative coalgebra,  $V$  is a vector space, and  $\alpha: C \rightarrow SV$  is a cocommutative coalgebra isomorphism. The morphisms of  $\mathcal{C}$  are coalgebra homomorphisms  $f: C \rightarrow C'$ . We will often denote  $(C, V, \alpha)$  by  $C \cong SV$ .*

Recall that since coalgebra homomorphisms send primitive elements to primitive elements,  $f: SV \rightarrow SV'$  automatically maps  $V$  to  $V'$ . Notice that this category has finite products, since it has binary products and a terminal object. That is, given  $(C, V, \alpha)$  and  $(C', V', \alpha')$ , their product is  $(C \otimes C', V \oplus V', (\alpha \otimes \alpha')\phi)$  where we have used the fact that the tensor product of cocommutative coalgebras is a cocommutative coalgebra and  $\phi: SV \otimes SV' \rightarrow S(V \oplus V')$  is the standard isomorphism. We remark that the cocommutativity of our special coalgebras is necessary for  $\mathcal{C}$  to have products. The terminal object in  $\mathcal{C}$  is  $(k, \{0\}, 1)$ .

We are now in a position to describe the functor  $J_\infty: \text{Diff}_* \rightarrow \mathcal{C}$ , since  $J_\infty(M, p)$  is a coalgebra that is isomorphic to  $S(T_p M)$ . This functor sends any object  $(M, p) \in \text{Diff}_*$  to  $(J_\infty(M, p), T_p(M), \psi)$  and any smooth map  $f: (M, p) \rightarrow (N, q)$  to  $J_\infty(f): J_\infty(M, p) \rightarrow J_\infty(N, q)$ . The functoriality and covariance of  $J_\infty$  follow from routine computations. Moreover,  $J_\infty$  preserves groups, and hence quandles in  $\mathcal{C}$ , since it preserves products:

**Proposition 98.** *The functor  $J_\infty: \text{Diff}_* \rightarrow \mathcal{C}$  preserves finite products.*

**Proof.** Given  $(M, p)$  and  $(N, q)$  in  $\text{Diff}_*$ , we must show that  $J_\infty(M \times N, (p, q))$  is the product of  $J_\infty(M, p)$  and  $J_\infty(N, q)$  in  $\mathcal{C}$ . We have

$$\begin{aligned} J_\infty(M, p) &:= (J_\infty(M, p), T_p M, \alpha) \\ J_\infty(N, q) &:= (J_\infty(N, q), T_q N, \alpha') \\ J_\infty(M \times N, (p, q)) &:= (J_\infty(M \times N, (p, q)), T_{(p, q)}(M \times N), \beta) \\ J_\infty(M, p) \times J_\infty(N, q) &:= (J_\infty(M, p) \otimes J_\infty(N, q), T_p M \oplus T_q N, (\alpha \otimes \alpha')\phi) \end{aligned}$$

We need only to demonstrate that  $J_\infty(M, p) \otimes J_\infty(N, q) \cong J_\infty(M \times N, (p, q))$ . By lemma 95,  $J_\infty(M, p) \cong S(T_p M)$ , so that we have

$$\begin{aligned} J_\infty(M, p) \otimes J_\infty(N, q) &\cong S(T_p M) \otimes S(T_q N) \\ &\cong S(T_p M \oplus T_q N) \\ &= S(T_{(p, q)}(M \times N)) \\ &\cong J_\infty(M \times N, (p, q)). \end{aligned}$$

It follows immediately that  $\beta \cong (\alpha \otimes \alpha')\phi$  since the following diagram commutes:

$$\begin{array}{ccc} J_\infty(M, p) \otimes J_\infty(N, q) & \xrightarrow{\alpha \otimes \alpha'} & S(T_p M) \otimes S(T_q N) \\ \downarrow \wr & & \downarrow \phi \\ J_\infty(M \times N, (p, q)) & \xrightarrow{\beta} & S(T_{(p, q)}(M \times N)) = S(T_p M \oplus T_q N) \end{array}$$

where the isomorphism on the left side of this square is the one described above.  $\square$

Since  $J_\infty$  preserves products, we obtain the following corollary:

**Corollary 99.** *The functor  $J_\infty: \text{Diff}_* \rightarrow \mathcal{C}$  sends groups in  $\mathcal{C}$  to groups in  $\mathcal{C}$ .*

Thus, we have completely described the first piece of our process:

$$\begin{array}{ccc} \text{Lie groups} & & \\ \downarrow & & \\ \text{Groups in } \text{Diff}_* & \xrightarrow{U} & \text{Diff}_* \\ \downarrow & & \downarrow J_\infty \\ \text{Groups in } \mathcal{C} & \xrightarrow{U} & \mathcal{C} \end{array}$$

Note that a group in the category of coalgebras is a Hopf algebra, and we are getting  $U\mathfrak{g} = J_\infty(G, 1)$ . Now, given a Lie group,  $(G, 1)$ , it remains to show how to peel off  $\mathfrak{g}$  from  $J_\infty(G, 1)$ .

### 3.2.3 Unital Spindles

We begin this section by reminding the reader of the connection between Lie algebras and spindles. Both Lie algebras and spindles are gadgets that give rise to Yang–Baxter operators. In Section 3.1.2 we saw that the Yang–Baxter equation is none other than the distributive law for the spindle operation, and that it is the Jacobi identity in disguise when we define our Yang–Baxter operator on the space  $k \oplus L$ , where  $L$  is a Lie algebra over  $k$ . Moreover, and most relevant to our task, we noticed how conjugation plays a prominent role in both the theory of spindles and that of Lie algebras. The operation of conjugation in groups satisfies the spindle axioms; indeed, groups were our primordial examples of spindles. Conjugation also appears at the heart of the theory of Lie algebras since the bracket in a Lie algebra arises from differentiating conjugation twice. Furthermore, the Jacobi identity in a Lie algebra arises from differentiating the self-distributivity axiom in a spindle when the spindle operation is conjugation. That is, given curves  $e^{sx}, e^{ty}$  and  $e^{uz}$  in a matrix Lie group  $G$  where  $x, y, z \in \mathfrak{g}$ , we have:

$$[x, [y, z]] = \frac{d}{dsdtdu} (e^{sx} e^{ty} e^{uz} e^{-ty} e^{-sx})|_{s=t=u=0}$$

while

$$[y, [x, z]] + [[x, y], z] = \frac{d}{dsdtdu} (e^{sx} e^{ty} e^{-sx} e^{sx} e^{uz} e^{-sx} e^{sx} e^{-ty} e^{-sx})|_{s=t=u=0}.$$

These observations taken together inspired the novel passage from Lie groups to Lie algebras that we continue describing. In this section, we describe the following aspect of our diagram:

$$\begin{array}{ccc} \text{Groups in } \mathcal{C} & \xrightarrow{U} & \mathcal{C} \\ \downarrow & & \downarrow 1 \\ \text{Unital Spindles in } \mathcal{C} & \xrightarrow{U} & \mathcal{C} \end{array}$$

Given a Lie group  $G$ , we will define an operation of conjugation on the coalgebra  $J_\infty(G, 1)$  to make it into a spindle. More precisely,  $J_\infty(G, 1)$  will be a spindle in  $\mathcal{C}$ . Recall that in Section 3.1.4 we demonstrated how a group in  $K$ , where  $K$  is a category with finite products, gives a quandle in  $K$ . This amounted to an internalization of the fact that usual groups give quandles. Actually we do not need the whole structure of a quandle to obtain a Lie algebra: it suffices to use left conjugation to define the bracket, so we can work with a spindle. But not any spindle object in  $\mathcal{C}$  gives a Lie algebra. This internalized spindle must possess two special properties, namely the internalized versions of:

$$x1x^{-1} = 1 \quad \text{and} \quad 1x1^{-1} = x,$$

where  $x \in G$  for some group  $G$  and  $1 \in G$  is the identity. We call a spindle with these two properties a ‘unital spindle’.

**Definition 100.** Let  $K$  be a category with finite products. A **unital spindle in  $K$**  is a spindle  $Q$  in  $K$  equipped with a special point  $\text{id}: I \rightarrow Q$ , where  $I$  is the terminal object in  $K$ , such that the following diagrams commute:

(a)

$$Q \xrightarrow{\quad} I \times Q \xrightarrow{\text{id} \times 1} Q \times Q \xrightarrow{\triangleright} Q$$

1

(b)

$$Q \xrightarrow{\quad} Q \times I \xrightarrow{1 \times \text{id}} Q \times Q \xrightarrow{\quad \triangleright \quad} Q$$

Since the two conditions given in the definition of a unital spindle are just the internalizations of the conjugation properties involving the identity element of a group, it should not come as a surprise that we can obtain a unital spindle in  $K$  from a group in  $K$ .

**Proposition 101.** *Let  $K$  be a category with finite products. A group  $Q$  in  $K$  gives a unital spindle in  $K$ .*

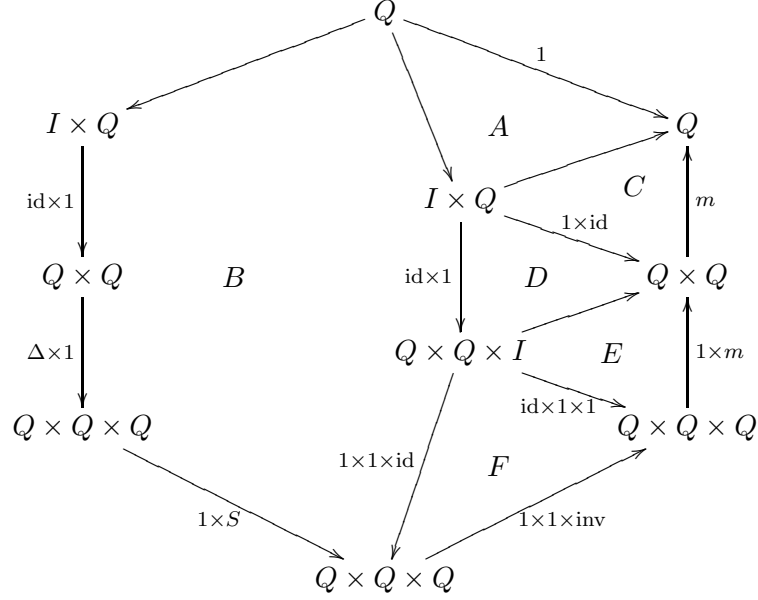
**Proof.** In Section 3.1.4, we began to illustrate how a group in  $K$  gives a quandle in  $K$ , and hence spindle in  $K$ , by defining the quandle operations in terms of conjugation. Since we are only concerned with spindles, we recall that we defined left conjugation  $\triangleright: Q \times Q \rightarrow Q$  as the composite:

$$(x, y) \xrightarrow{\Delta \times 1} (x, x, y) \xrightarrow{1 \times S} (x, y, x) \xrightarrow{1 \times 1 \times \text{inv}} (x, y, x^{-1}) \xrightarrow{1 \times m} (x, yx^{-1}) \xrightarrow{m} xyx^{-1}$$

We then demonstrated how to obtain the internalized left idempotence law. Showing that the internalized left distributive law can be obtained by replacing  $\triangleright$  in diagram (i') of Definition 75 amounts to a rather large, though mostly trivial, diagram.

Obtaining the two additional properties of a unital quandle require showing that the identity in our group object,  $\text{id}: I \rightarrow Q$ , makes the diagrams (a) and (b) above commute, where  $\triangleright$  is replaced by the composite given above.

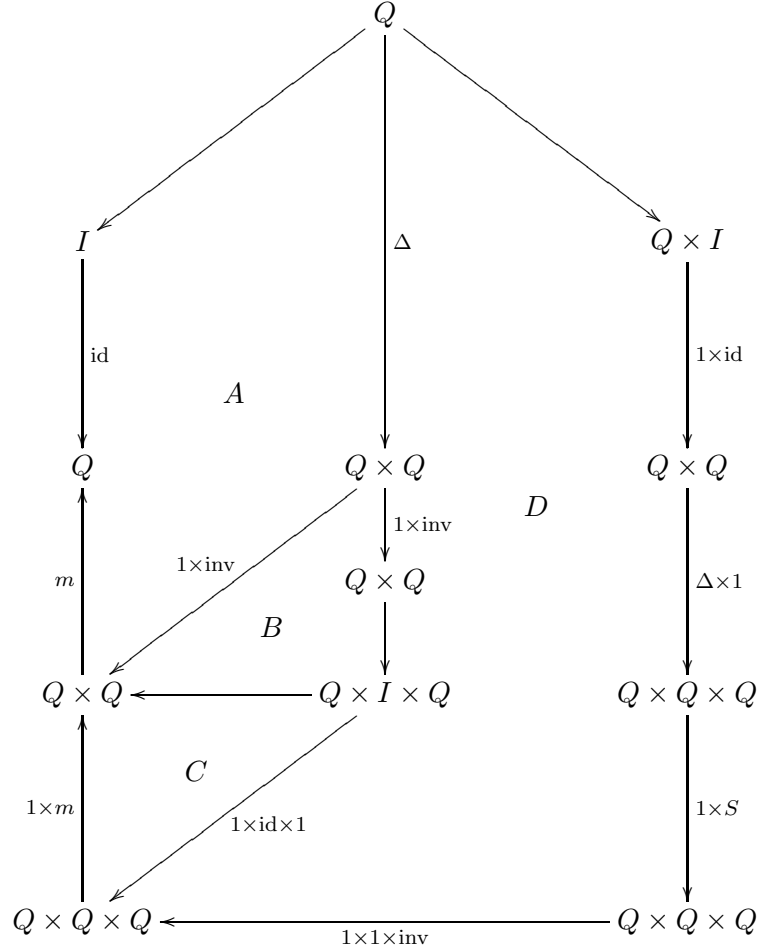
Property (a) is the result of the diagram:



where region  $C$  is the left unit law, region  $E$  is the right unit law, region  $F$  is the internalization of the fact that the inverse of the identity of a group is the identity. Regions  $A$ ,  $B$  and  $D$  clearly commute.



Property (b) is given by the diagram:



where region  $A$  is the right inverse law and region  $C$  is the left unit law. Regions  $B$  and  $D$  clearly commute. Thus we have that a group in  $K$  gives a unital spindle in  $K$ .  $\square$

We will now use the result of Proposition 101 to turn our groups in  $\mathcal{C}$  into unital spindles in  $\mathcal{C}$ . We remark that while we have chosen to use the less familiar notion of spindle for this process, it is also true that a unital quandle in  $\mathcal{C}$  will give a Lie algebra. Moreover, Lie groups actually give unital quandles in  $\mathcal{C}$ , where a unital quandle is merely a unital spindle which is a quandle.

It remains, then, to describe the final aspect of our process: how to obtain a Lie algebra from a unital spindle in  $\mathcal{C}$ . Thus, we need only to define the functor  $F: \mathcal{C} \rightarrow \text{Vect}$  and show that when we start with a unital spindle  $Q \cong SV$  for some vector space  $V$ , that we can define a bracket operation on  $V$  making it into a Lie algebra.

### 3.2.4 From Unital Spindles to Lie Algebras

In this section, we finish describing the process of obtaining a Lie algebra from a Lie group using the language of spindles. That is, we explain this final aspect of our

diagram:

$$\begin{array}{ccc}
 \text{Unital Spindles in } \mathcal{C} & \xrightarrow{U} & \mathcal{C} \\
 \downarrow & & \downarrow F \\
 \text{Lie algebras} & \xrightarrow{U} & \text{Vect}
 \end{array}$$

Given a special coalgebra  $C \cong SV$  for some vector space  $V$ , we define  $F: \mathcal{C} \rightarrow \text{Vect}$  to send any object  $(C, V, \alpha)$  to  $V$  and any morphism  $f: C \rightarrow C'$  to  $\alpha^{-1} \circ f \circ \alpha|_V: V \rightarrow V'$ .

Now, in order to show that a unital spindle  $Q$  in  $\mathcal{C}$  gives a Lie algebra, it is beneficial to know how to interpret conditions (a) and (b) in Definition 100 for an element  $x \in Q$ . To do so, we need the following lemmas:

**Lemma 102.** *If  $C$  is a special coalgebra, the counit  $\epsilon_C: C \rightarrow k$  is the unique homomorphism to the terminal object  $I$  in  $\mathcal{C}$ .*

**Proof.** Recall that the terminal object,  $I$ , in  $\mathcal{C}$  is given by  $(k, \{0\}, 1)$ . Thus, by definition of a morphism in  $\mathcal{C}$ , a morphism from the special coalgebra  $(C, V, \alpha)$  to the terminal object  $(k, \{0\}, 1)$  is a coalgebra homomorphism  $f: C \rightarrow k$ . Therefore, we must show  $\epsilon_C: C \rightarrow k$  is a coalgebra homomorphism, meaning we must show that these two diagrams:

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta_C} & C \otimes C \\
 \downarrow \epsilon_C & & \downarrow \epsilon_C \otimes \epsilon_C \\
 k & \xrightarrow{\Delta_k} & k \otimes k
 \end{array}$$

$$\begin{array}{ccc}
 C & \xrightarrow{\epsilon_C} & k \\
 \searrow \epsilon_C & & \uparrow \epsilon_k \\
 & & k
 \end{array}$$

commute, which is clear.  $\square$

**Lemma 103.** *If  $C$  is a special coalgebra, the comultiplication  $\Delta_C: C \rightarrow C \otimes C$  is the diagonal map in  $\mathcal{C}$ .*

**Proof.** Since  $C \otimes C$  is the product in  $\mathcal{C}$ , there is a unique morphism  $\phi: C \rightarrow C \otimes C$  which makes the following diagram commute:

$$\begin{array}{ccccc}
 & & C & & \\
 & \swarrow 1_C & \downarrow \phi & \searrow 1_C & \\
 C & \xleftarrow{\pi_1} & C \otimes C & \xrightarrow{\pi_2} & C
 \end{array}$$

where the  $\pi_1$  and  $\pi_2$  are projection maps. This map  $\phi$  is called the diagonal. Since  $C$  is a coalgebra, we have the comultiplication map  $\Delta_C: C \rightarrow C \otimes C$ . This makes the same sort of diagram commute:

$$\begin{array}{ccccc}
 & & C & & \\
 & \swarrow 1_C & \downarrow \Delta_C & \searrow 1_C & \\
 C & \xleftarrow{\pi_1} & C \otimes C & \xrightarrow{\pi_2} & C
 \end{array}$$

since the projection maps are the following composites:

$$\begin{aligned}
 \pi_1 &:= A \otimes B \xrightarrow{1_A \otimes \epsilon_B} A \otimes k \xrightarrow{\sim} A \\
 \pi_2 &:= A \otimes B \xrightarrow{\epsilon_A \otimes 1_B} k \otimes B \xrightarrow{\sim} B
 \end{aligned}$$

where  $\epsilon_A$  and  $\epsilon_B$  are the counit maps for coalgebras  $A$  and  $B$ . Since the diagonal map  $\phi$  is unique, it must equal  $\Delta$ .  $\square$

**Lemma 104.** *If  $(C, V, \alpha)$  is a special coalgebra with the structure of a unital spindle in  $\mathcal{C}$ , then the map  $\text{id}: k \rightarrow SV$ , obtained by composing  $\text{id}: k \rightarrow C$  with the isomorphism  $\alpha: C \rightarrow SV$ , maps the multiplicative identity 1 in  $k$  to the multiplicative identity 1 in  $SV$ . That is,  $\text{id}(1) = 1$ .*

**Proof.** Notice that in the statement of this result, we took advantage of Lemma 102 so that we could refer to the terminal object of  $\mathcal{C}$  as  $k$ .

To prove this result, note first that since  $1 \in k$  satisfies  $\Delta(1) = 1 \otimes 1$  and  $\text{id}: k \rightarrow SV$  is a coalgebra homomorphism, we have  $\Delta(\text{id}(1)) = \text{id}(1) \otimes \text{id}(1)$ . Thus, it suffices to show that the only element  $x \in SV$  with  $\Delta(x) = x \otimes x$  is  $x = 1$ .

To do this, note that we can think of  $SV$  as the algebra of polynomial functions on  $V^*$ , and the comultiplication  $\Delta: SV \rightarrow SV \otimes SV \cong S(V \oplus V)$  as the following map from polynomials on  $V$  to polynomials on  $V^* \oplus V^*$ :

$$\Delta(x)(a, b) = x(a + b)$$

for all  $a, b \in V^*$  and all polynomials  $x$  on  $V^*$ . The condition that  $\Delta(x) = x \otimes x$  thus says that

$$x(a + b) = x(a)x(b)$$

for all  $a, b \in V^*$ . If  $x$  is a polynomial of degree  $n$  on  $V^*$ , the left side of the above equation describes a polynomial of degree  $n$  on  $V^* \oplus V^*$ , while the right side describes a polynomial of degree  $2n$ . This is only possible if  $n = 0$ , so  $x$  must be a constant polynomial, and the only constant polynomial satisfying the above equation is  $x = 1$ .  $\square$

Lemma 104 together with Lemma 102 allow us to interpret conditions (a) and (b) in Definition 100 for an element  $x \in Q$ , where  $Q$  is a unital spindle in  $\mathcal{C}$ . Axiom (a) becomes

$$1 \triangleright x = x,$$

while axiom (b) becomes

$$x \triangleright 1 = \epsilon(x)1.$$

We will make use of both of these identities below.

We now have all the ingredients to construct a Lie algebra from a unital spindle in  $\mathcal{C}$ . Proposition 101 demonstrated that groups in  $\mathcal{C}$  give unital spindles in  $\mathcal{C}$ . Thus, given a group in  $\mathcal{C}$ , it becomes a unital spindle in  $\mathcal{C}$ , say  $(Q, V, \alpha)$ . Now we shall use the spindle operation  $\triangleright: Q \otimes Q \rightarrow Q$  to define a bracket on  $V$ . Since we want a map  $[\cdot, \cdot]: V \otimes V \rightarrow V$ , we first use the isomorphism  $\alpha$  to transfer  $\triangleright$  from  $Q$  to  $SV$ :

$$\begin{array}{ccc} SV \otimes SV & \xrightarrow{\triangleright} & SV \\ \alpha^{-1} \otimes \alpha^{-1} \downarrow & & \uparrow \alpha \\ Q \otimes Q & \xrightarrow{\triangleright} & Q \end{array}$$

We then define the bracket as the following composite:

$$\begin{array}{ccc} V \otimes V & \xrightarrow{[\cdot, \cdot]} & V \\ \downarrow & & \uparrow \pi \\ SV \otimes SV & \xrightarrow{\triangleright} & SV \end{array}$$

Using this, we finally obtain our desired result:

**Theorem 105.** *Given a unital spindle  $Q \cong SV$  in the category  $\mathcal{C}$  of special coalgebras, and defining  $[\cdot, \cdot]: V \otimes V \rightarrow V$  as*

$$\begin{array}{ccc} V \otimes V & \xrightarrow{[\cdot, \cdot]} & V \\ \downarrow & & \uparrow \pi \\ SV \otimes SV & \xrightarrow{\triangleright} & SV \end{array}$$

*then  $V$  becomes a Lie algebra.*

**Proof.** To prove the antisymmetry of the bracket it suffices to show that  $[v, v] = 0$  for all  $v \in V$ , and by the above definition of the bracket it suffices to show that  $v \triangleright v = 0$ . We will

prove this using the internalized version of the left idempotence law of a spindle. Recall that the left idempotence law says that

$$\begin{array}{ccc} & & SV \times SV \\ & \nearrow \Delta & \downarrow \triangleright \\ SV & \xrightarrow{1} & SV \end{array}$$

commutes, where  $SV \times SV$  is computed using the product in  $\mathcal{C}$ , namely the tensor product of special coalgebras.

Let  $v \in V$ . Then  $v^2 \in S(V) \cong Q$ . Applying the internalized idempotence law to  $v^2$ , we have

$$v^2 = \triangleright(\Delta(v^2)) = \triangleright(\Delta(v)\Delta(v)),$$

since  $\Delta$  is an algebra homomorphism, due to the fact that  $SV$ , and hence  $Q$ , is a bialgebra. But by Lemma 103, the diagonal  $\Delta$  is the same as the comultiplication map on  $Q$ , so we have:

$$\begin{aligned} \triangleright(\Delta(v)\Delta(v)) &= \triangleright[(v \otimes 1 + 1 \otimes v)(v \otimes 1 + 1 \otimes v)] \\ &= \triangleright(v^2 \otimes 1 + v \otimes v + v \otimes v + 1 \otimes v^2) \\ &= v^2 \triangleright 1 + 2v \triangleright v + 1 \triangleright v^2. \end{aligned}$$

and thus

$$v^2 = v^2 \triangleright 1 + 2v \triangleright v + 1 \triangleright v^2.$$

Now we recall that conditions (a) and (b) of a unital spindle say, respectively, that

$$1 \triangleright x = x \quad \text{and} \quad x \triangleright 1 = \epsilon(x)1$$

for  $x \in Q$ . We thus have

$$v^2 = \epsilon(v^2) + 2v \triangleright v + v^2$$

but the counit in  $SV$  satisfies  $\epsilon(v^2) = 0$ , so

$$v \triangleright v = 0$$

as desired.

Similarly, the Jacobi identity follows from the fact that

$$u \triangleright (v \triangleright w) = (u \triangleright v) \triangleright w + v \triangleright (u \triangleright w)$$

for all  $u, v, w \in V$ . This in turn follows from the internalized left distributive law of our

spindle in  $\mathcal{C}$ :

$$\begin{array}{ccccc}
& & SV \times SV \times SV & & \\
& \swarrow \Delta \times 1 \times 1 & & \searrow 1 \times \triangleright & \\
SV \times SV \times SV \times SV & & & & SV \times SV \\
\downarrow 1 \times S \times 1 & & & & \downarrow \triangleright \\
SV \times SV \times SV \times SV & & & & SV \\
& \searrow \triangleright \times 1 \times 1 & & \nearrow \triangleright & \\
& SV \times SV \times SV & \xrightarrow{1 \times \triangleright} & SV \times SV &
\end{array}$$

which says

$$\begin{array}{ccccc}
& & u \otimes v \otimes w & & \\
& \swarrow \Delta \otimes 1 \otimes 1 & & \searrow 1 \otimes \triangleright & \\
(u \otimes 1 + 1 \otimes u) \otimes v \otimes w & & & & u \otimes [v, w] \\
= u \otimes 1 \otimes v \otimes w + 1 \otimes u \otimes v \otimes w & & & & \downarrow \triangleright \\
\downarrow 1 \otimes S \otimes 1 & & & & u \triangleright (v \triangleright w) = \\
u \otimes v \otimes 1 \otimes w + 1 \otimes v \otimes u \otimes w & & & & (u \triangleright v) \triangleright w + v \triangleright (u \triangleright w) \\
& \searrow \triangleright \otimes 1 \otimes 1 & & \nearrow \triangleright & \\
& u \triangleright v \otimes 1 \otimes w + v \otimes u \otimes w & \xrightarrow{1 \otimes \triangleright} & u \triangleright v \otimes w + v \otimes u \triangleright w &
\end{array}$$

Thus, unital spindles in  $\mathcal{C}$  give Lie algebras.  $\square$

But, our original goal was to obtain the Lie algebra of a given Lie group, not of a unital spindle in  $\mathcal{C}$ ! Applying the previous theorem to the special case when the unital spindle in  $\mathcal{C}$  came from a group in  $\mathcal{C}$ , which, in turn, came from a group in  $\text{Diff}_*$ , we obtain the result we desire.

To summarize, we carry out the following procedure: We begin by using the language of internalization to say that a Lie group  $G$  is a group in  $\text{Diff}_*$ . We next apply the cojet functor  $J_\infty$ , obtaining a group in  $\mathcal{C}$  since this functor preserves products. A group in  $\mathcal{C}$  automatically gives a unital spindle in  $\mathcal{C}$ . Then, applying the functor  $F$  to this result, we extract a vector space which we call  $\mathfrak{g}$ . Finally, as above, we define the bracket on this

vector space  $\mathfrak{g}$  as:

$$\begin{array}{ccc}
 \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{[\cdot, \cdot]} & \mathfrak{g} \\
 \downarrow & & \uparrow \pi \\
 S\mathfrak{g} \otimes S\mathfrak{g} & \xrightarrow{\triangleright} & S\mathfrak{g}
 \end{array}$$

and use the axioms of a unital spindle in  $\mathcal{C}$  to show that this bracket operation is antisymmetric and satisfies the Jacobi identity, so that  $\mathfrak{g}$  is the Lie algebra of  $G$ , as desired.

**Corollary 106.** *If  $G$  is a Lie group and  $Q$  is the unital spindle in  $\mathcal{C}$  defined as described in Proposition 101, the Lie algebra  $\mathfrak{g}$  constructed in Theorem 105 is isomorphic to the Lie algebra of  $G$ .*

Thus, we have described a novel method of obtaining the Lie algebra of a Lie group using spindles. In the next section, we indicate how we anticipate categorifying this procedure in order to acquire the Lie 2-algebra of a Lie 2-group.

### 3.3 Lie 2-algebras, 2-Quandles and 2-Braids

We conclude by outlining how we suspect the categorification of the previous three sections will occur. We start by categorifying the notions of shelf, rack, spindle and quandle. Recall that categorifying a mathematical concept involves finding category-theoretic analogs of set-theoretic concepts. Therefore, to obtain a categorified shelf, or ‘2-shelf’, we will replace the set  $Q$  in Definition 52 by a category, the function  $\triangleright: Q \times Q \rightarrow Q$  by a functor, called ‘left conjugation’ and the equation (i) by a natural isomorphism, called the ‘left distributor’, which will take us from one side of the left distributive law to the other. In addition, we will require that the left distributor satisfy an equation of its own, known as a coherence law. While we only present the definition of a ‘2-shelf’ here, we hint at the correct formulations of the notions of ‘2-rack’, ‘2-spindle’ and ‘2-quandle’.

**Definition 107.** *A left 2-shelf,  $(Q, \triangleright)$ , consists of:*

- *a category  $Q$ ,*

*equipped with:*

- *a functor, left conjugation,  $\triangleright: Q \times Q \rightarrow Q$ ,*
- *a natural isomorphism, the left distributor,*

$$LD_{x,y,z}: x \triangleright (y \triangleright z) \rightarrow (x \triangleright y) \triangleright (x \triangleright z)$$

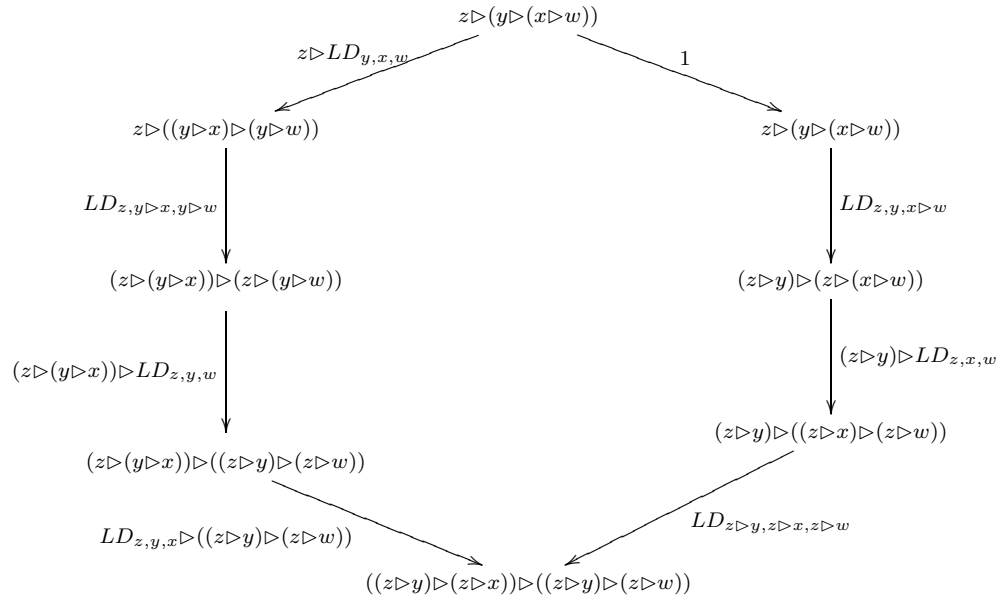
*that is required to satisfy:*

• *the distributor identity:*

$$(z \triangleright LD_{y,x,w}) LD_{z,y \triangleright x, y \triangleright w} ((z \triangleright (y \triangleright x)) \triangleright LD_{z,y,w}) (LD_{z,y,x} \triangleright ((z \triangleright y) \triangleright (z \triangleright w))) = \\ LD_{z,y,x \triangleright w} ((z \triangleright y) \triangleright LD_{z,x,w}) LD_{z \triangleright y, z \triangleright x, z \triangleright w}$$

for all  $w, x, y, z \in Q$ .

As in the case of the Jacobiator identity, the left distributor identity becomes more manageable if we draw it as a commutative diagram. Doing so enables us to see that this law relates two ways of using the left distributor to reparenthesize the expression  $z \triangleright (y \triangleright (x \triangleright w))$ :



At this point, we remark that we can take a cue from topology to assist us in our categorification! We saw in Section 3.1.2 that the Yang–Baxter equation, or third Reidemeister move, is equivalent to the left distributive law for the shelf operation. Therefore, rather than thinking of the left distributor solely as an algebraic object, we can think of it as the process of performing the third Reidemeister move. With this in mind, the coherence law for the left distributor will be the higher dimensional analogue of the third Reidemeister move, the Zamolodchikov tetrahedron equation, familiar from the theory of 2-knots and braided monoidal 2-categories [BLan, BN, CS, C, KV]. This equation plays a role in the theory of knotted surfaces in 4-space which is closely analogous to that played by the Yang–Baxter equation, or third Reidemeister move, in the theory of ordinary knots in 3-space.

The analogue of the Yang–Baxter equation is called the ‘Zamolodchikov tetrahedron equation’:

**Definition 108.** *Given a category  $C$  and an invertible functor  $B: C \times C \rightarrow C \times C$ , a natural isomorphism*

$$Y: (B \times 1)(1 \times B)(B \times 1) \Rightarrow (1 \times B)(B \times 1)(1 \times B)$$

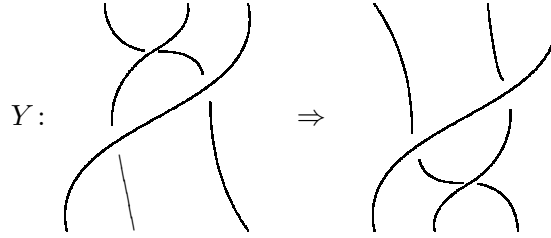


satisfies the **Zamolodchikov tetrahedron equation** if

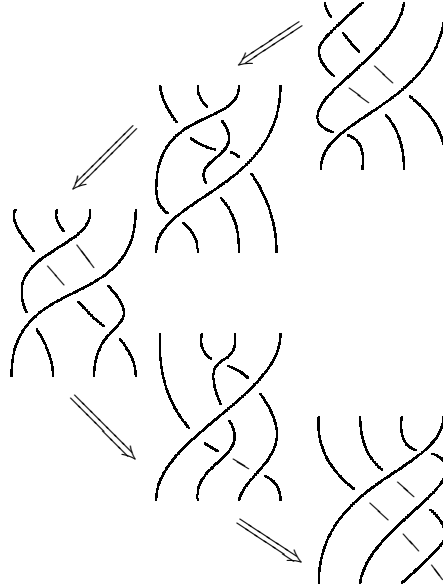
$$\begin{aligned}
& [Y \circ (1 \times 1 \times B)(1 \times B \times 1)(B \times 1 \times 1)][(1 \times B \times 1)(B \times 1 \times 1) \circ Y \circ (B \times 1 \times 1)] \\
& [(1 \times B \times 1)(1 \times 1 \times B) \circ Y \circ (1 \times 1 \times B)][Y \circ (B \times 1 \times 1)(1 \times B \times 1)(1 \times 1 \times B)] \\
& = \\
& [(B \times 1 \times 1)(1 \times B \times 1)(1 \times 1 \times B) \circ Y][(B \times 1 \times 1) \circ Y \circ (B \times 1 \times 1)(1 \times B \times 1)] \\
& [(1 \times 1 \times B) \circ Y \circ (1 \times 1 \times B)(1 \times B \times 1)][(1 \times 1 \times B)(1 \times B \times 1)(B \times 1 \times 1) \circ Y],
\end{aligned}$$

where  $\circ$  represents the whiskering of a functor by a natural transformation. We will often refer to  $Y$  as a **Yang–Baxterator**.

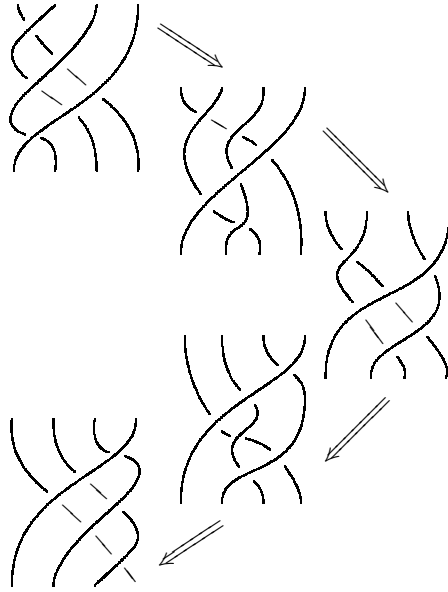
To see the significance of this complex but beautifully symmetrical equation, one should think of  $Y$  as the surface in 4-space traced out by the process of performing the third Reidemeister move:



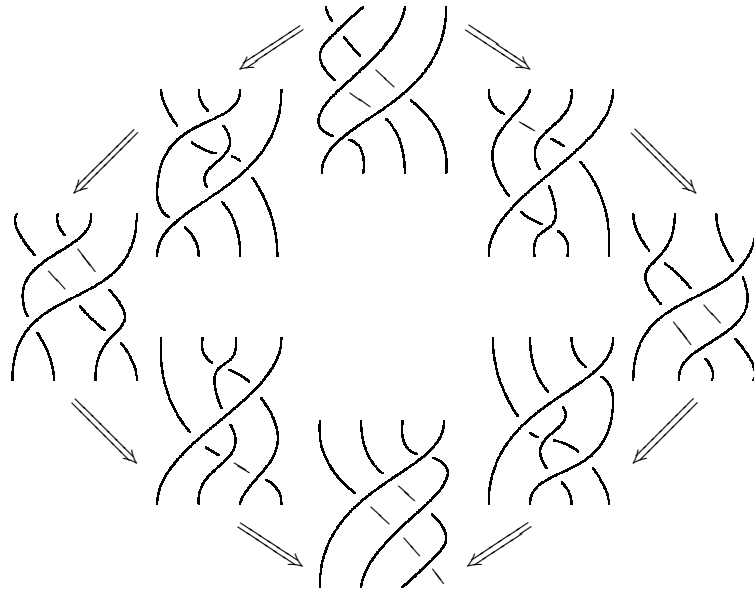
Then the Zamolodchikov tetrahedron equation says the surface traced out by first performing the third Reidemeister move on a threefold crossing and then sliding the result under a fourth strand:



is isotopic to that traced out by first sliding the threefold crossing under the fourth strand and then performing the third Reidemeister move:



In short, the Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



in a 2-category whose 2-morphisms are isotopies of surfaces in 4-space — or more precisely, ‘2-braids’. Details can be found in HDA1, HDA4 and a number of other references, going back to the work of Kapranov and Voevodsky [BLan, BN, CaS, C, KV].

As suggested above, we now see a relationship between the algebraic version of the coherence law for the left distributor and the Zamolodchikov tetrahedron equation! They are both octagons, which turn out to be equivalent in the following context:

**Theorem 109.** Let  $Q$  be a category equipped with a functor  $\triangleright: Q \times Q \rightarrow Q$  and a natural isomorphism

$$LD_{x,y,z}: x \triangleright (y \triangleright z) \rightarrow (x \triangleright y) \triangleright (x \triangleright z).$$

Define  $B: Q \times Q \rightarrow Q \times Q$  by

$$B(x, y) = (y, y \triangleright x)$$

whenever  $x$  and  $y$  are both either objects or morphisms in  $Q$ . Let

$$Y: (B \times 1)(1 \times B)(B \times 1) \Rightarrow (1 \times B)(B \times 1)(1 \times B)$$

be defined as:

$$Y_{x,y,z} = 1_z \times 1_{z \triangleright y} \times LD_{z,y,x}.$$

Then  $Y$  is a solution of the Zamolodchikov tetrahedron equation if and only if  $(Q, \triangleright, LD)$  is a left 2-shelf.

**Proof.** Applying the left-hand side of the Zamolodchikov tetrahedron equation to an object  $(w, x, y, z)$  of  $Q \times Q \times Q \times Q$  produces an expression consisting of various uninteresting terms together with one involving

$$(z \triangleright LD_{y,x,w})LD_{z,y \triangleright x, y \triangleright w}((z \triangleright (y \triangleright x)) \triangleright LD_{z,y,w})(LD_{z,y,x} \triangleright ((z \triangleright y) \triangleright (z \triangleright w)))$$

while applying the right-hand side gives an expression with the same uninteresting terms, but also one with

$$LD_{z,y,x \triangleright w}((z \triangleright y) \triangleright LD_{z,x,w})LD_{z \triangleright y, z \triangleright x, z \triangleright w}$$

in exactly the same way. Thus, the two sides are equal if and only if the left distributor identity holds.  $\square$

Therefore, we can think of the left distributor and its coherence law from a completely topological viewpoint!

The definition of a right 2-shelf is similar, but uses the versions of the third Reidemeister move and Zamolodchikov equation having left-handed crossings. Thus far, this topological connection has proven to be a useful guiding light in formulating the correct definitions of 2-shelves. As we move to racks and quandles, however, the situation becomes significantly more complex. The presence of the *two* conjugation operations complicates matters since each operation corresponds to its own type of crossing. We have seen that left conjugation corresponds to a positive, or right-handed, crossings, while right conjugation corresponds to a negative, or left-handed, crossings.

Thus, a 2-rack will consist of a left and right 2-shelf equipped with two inverse natural isomorphisms

$$L_{x,y}: x \triangleright (y \triangleleft x) \rightarrow x \quad R_{x,y}: (y \triangleright x) \triangleleft y \rightarrow x$$

which we can draw as:

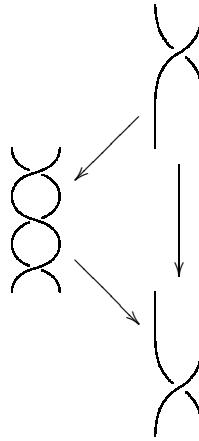
$$\begin{array}{c} \text{Crossing} \end{array} \longrightarrow \begin{array}{c} \text{Two parallel lines} \end{array} \quad R_{x,y}: (y \triangleright x) \triangleleft y \Rightarrow x$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \longrightarrow \begin{array}{c} | \\ | \end{array} \qquad L_{x,y}: x \triangleright (y \triangleleft x) \Rightarrow y$$

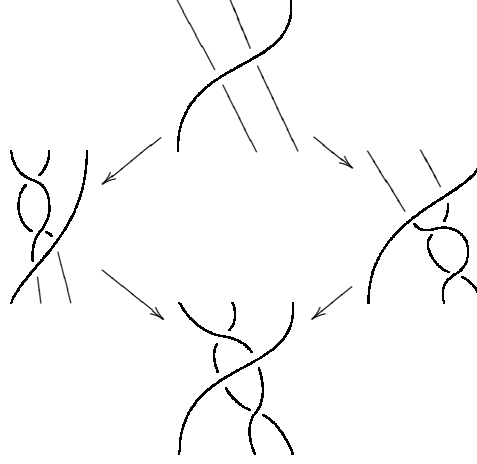
that are required to satisfy additional coherence laws. Due to the presence of the two conjugation functors corresponding to two different crossing changes, describing all of these coherence laws is no easy task. This is a consequence of the fact that these two conjugation functors allow for  $2^3 = 8$  possible ways of performing the third Reidemeister move, corresponding to the two choices of crossing for each of the three crossings of one side of the third Reidemeister move. Thus, a priori, we should expect 8 different versions of  $Y$ . However, drawing pictures of each of these higher Yang–Baxter operators reveals that 2 are impossible, since the strands become too tangled up. That is, there is no version of the Yang–Baxter equation with either  $(B \times 1)(1 \times B^{-1})(B \times 1)$  or  $(B^{1-} \times 1)(1 \times B)(B^{-1} \times 1)$  on the left-hand side, regardless of what you try for the right-hand side. These six versions of  $Y$ , then suggest at most  $2^6 = 64$  versions of the Zamolodchikov tetrahedron equation corresponding to the two choices of crossing for each of the six crossings in the top braid of the equation. Fortunately, we have shown that, of these six, there is only one independent  $Y$ . That is, we are able to derive any of the other five from the  $Y$  related to a left 2-shelf described above. However, this does not mean that we have now reduced the number of independent Zamolodchikov tetrahedron equations down to one. We certainly can express each of the Zamolodchikov tetrahedron equations in terms of our favorite  $Y$ , but it is not immediately obvious how many remain.

In addition to the Zamolodchikov tetrahedron equation, we anticipate the natural isomorphisms of a 2-rack to satisfy coherence laws relating the untanglers to the distributors. That is, we expect to require the commutativity of various versions of the following two diagrams:

(i)



(ii)

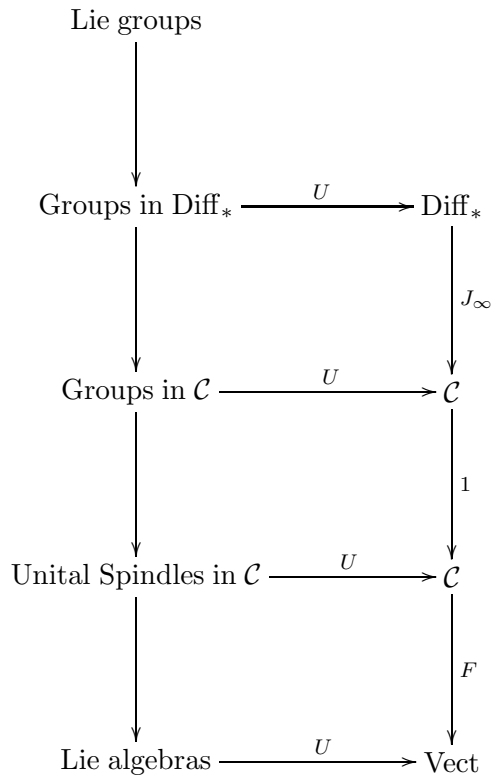


We have not yet determined the number of independent coherence laws which arise from using both left and right-handed crossings in the two diagrams above.

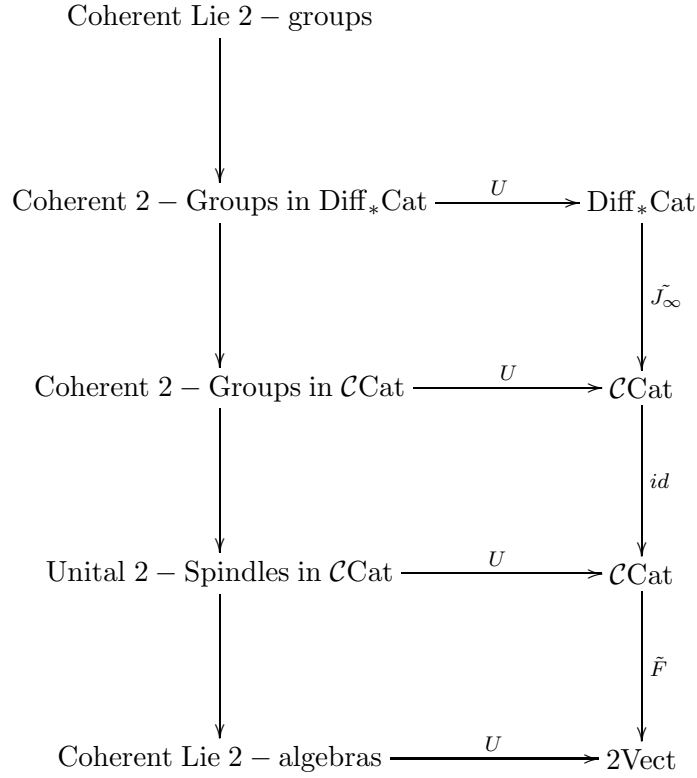
Since we have already begun to wade into a sea of uncertainty, and since 2-quandles are equipped with still more natural isomorphisms, we will end our discussion of the categorifications of these four algebraic objects. Before we move on, however, we remark that we expect that the coherent 2-groups described in HDA5 [BLau] will give examples of 2-quandles just as groups provide examples of quandles. Moreover, just as Lie algebra cohomology was used to classify semistrict Lie 2-algebras in Section 2.5, we anticipate that quandle and rack cohomology [CJKLS, EG] will serve to classify 2-quandles and 2-racks. Finally, we hope to prove a categorified version of Theorem 74 from Section 3.1.3, since the categorified versions of the braid and framed braid groups and monoids are known. This categorified result will give an invariant of 2-braids.

Ultimately, we desire to demonstrate that every Lie 2-group has a Lie 2-algebra, using a categorified version of the process described in Section 3.2. Recall that this process

was described by the following diagram:



One might hope that the passage from Lie 2-group to Lie 2-algebra will be described by a diagram in which we have categorified everything in sight:



The good news is that we already have a definition of a coherent 2-group in  $K$ , where  $K$  is a 2-category with finite products, which then gives us the notion of coherent Lie 2-group. Moreover, we know what a 2-vector space is, and that  $\text{DiffCat}$  and  $K\text{Cat}$  have finite products since  $\text{Diff}$  and  $K$  do. This is promising because it means that the notion of a 2-spindle in one of these 2-categories makes sense.

Unfortunately, it seems the bad news outweighs the good. As mentioned at the end of Section 2.4 of Chapter 2, this diagram will probably require what we call ‘coherent’ Lie 2-algebras, in which the antisymmetry law of the bracket is replaced by a natural isomorphism which we call the ‘antisymmetrizer’, but where bilinearity continues to hold on the nose. We suspect that such Lie 2-algebras are necessary because there are various laws that hold only up to isomorphism in a coherent Lie 2-group, most likely exactly the laws that we need to get skew-symmetry of the bracket via differentiation.

We have already begun to investigate the result of weakening the skew-symmetry condition by introducing an additional natural transformation into the definition of a semistrict Lie 2-algebra called the **antisymmetrizer**:

$$A_{x,y}: [x, y] \rightarrow -[y, x],$$

which, together with its coherence laws will give the definition of a ‘coherent’ Lie 2-algebra. Thus far we have found four coherence laws for the antisymmetrizer, three of which relate

it to the Jacobiator. Though we do not, yet, have a complete definition of a 2-quandle, we know in principle what it should be like, and we have determined some of its coherence laws. At this point, it is interesting to remark that we were only able to derive two of the coherence laws for a coherent Lie 2-algebra from two of the known coherence laws of a 2-quandle. Therefore, the other two laws, including

$$\begin{array}{ccc}
 & [[x, y], z] & \\
 J_{x,y,z} \swarrow & & \nwarrow [x, A_{y,z}] \\
 [x, [y, z]] + [[x, z], y] & \xrightarrow{J_{x,z,y}} & [x, [y, z]] + [x, [z, y]] + [[x, y], z]
 \end{array}$$

appear to be unrelated to topology, in particular, to the 2-braid diagrams. If so, this suggests that our strategy for passing from coherent Lie 2-groups to coherent Lie 2-algebras via 2-spindles may be too naive. We hope that after this categorified version is understood, we will be able to show that a coherent Lie 2-algebra gives a 2-spindle in  $\mathcal{CCat}$  in analogy to Theorem 105.

We conclude with evidence supporting our conjectures relating categorified Lie theory to higher-dimensional topology, as a way of showing that our guesses are not too far afield. We show that, in a suitable context, the Jacobiator identity is equivalent to the Zamolodchikov tetrahedron equation! However, we need a different version of the Zamolodchikov tetrahedron equation than the one given earlier in this section. This is because we defined a 2-shelf in the 2-category  $\mathcal{Cat}$ , which is a monoidal 2-category where the tensor product is the Cartesian product, whereas the definition of a Lie 2-algebra lives in the 2-category  $2\mathbf{Vect}$ , which is a monoidal 2-category with the usual tensor product. Thus, we need the following:

**Definition 110.** *Given a 2-vector space  $V$  and an invertible linear functor  $B: V \otimes V \rightarrow V \otimes V$ , a linear natural isomorphism*

$$Y: (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

*satisfies the **Zamolodchikov tetrahedron equation** if*

$$\begin{aligned}
 & [Y \circ (1 \otimes 1 \otimes B)(1 \otimes B \otimes 1)(B \otimes 1 \otimes 1)][(1 \otimes B \otimes 1)(B \otimes 1 \otimes 1) \circ Y \circ (B \otimes 1 \otimes 1)] \\
 & [(1 \otimes B \otimes 1)(1 \otimes 1 \otimes B) \circ Y \circ (1 \otimes 1 \otimes B)][Y \circ (B \otimes 1 \otimes 1)(1 \otimes B \otimes 1)(1 \otimes 1 \otimes B)] \\
 & = \\
 & [(B \otimes 1 \otimes 1)(1 \otimes B \otimes 1)(1 \otimes 1 \otimes B) \circ Y][(B \otimes 1 \otimes 1) \circ Y \circ (B \otimes 1 \otimes 1)(1 \otimes B \otimes 1)] \\
 & [(1 \otimes 1 \otimes B) \circ Y \circ (1 \otimes 1 \otimes B)(1 \otimes B \otimes 1)][(1 \otimes 1 \otimes B)(1 \otimes B \otimes 1)(B \otimes 1 \otimes 1) \circ Y],
 \end{aligned}$$

*where  $\circ$  represents the whiskering of a linear functor by a linear natural transformation.*



Though we have seen that coherent Lie 2-algebras may not be as closely related to topology as we would like, it is the case, however, that semistrict Lie 2-algebras have a connection to higher-braid theory. Recall from Section 2.3.1 that the coherence law for the Jacobiator, the Jacobiator identity in Definition 22, seems rather arcane. It turns out to be related to the Zamolodchikov tetrahedron equation in  $2\text{Vect}$  in the same way that the Jacobi identity is related to the Yang–Baxter equation. That is, Proposition 64, which roughly stated that a Lie algebra gives a solution of the Yang–Baxter equation, has a higher-dimensional analog, obtained by categorifying everything in sight! Recall in the definition of a semistrict Lie 2-algebra, Definition 22, we clarified the Jacobiator identity by drawing it as a commutative octagon. In fact, that commutative octagon becomes *equivalent* to the octagon for the Zamolodchikov tetrahedron equation given earlier in this section in the following context:

**Theorem 111.** *Let  $L$  be a 2-vector space, let  $[\cdot, \cdot]: L \times L \rightarrow L$  be a skew-symmetric bilinear functor, and let  $J$  be a completely antisymmetric trilinear natural transformation with  $J_{x,y,z}: [[x, y], z] \rightarrow [x, [y, z]] + [[x, z], y]$ . Let  $L' = K \oplus L$ , where  $K$  is the categorified ground field. Let  $B: L' \otimes L' \rightarrow L' \otimes L'$  be defined as follows:*

$$B((a, x) \otimes (b, y)) = (b, y) \otimes (a, x) + (1, 0) \otimes (0, [x, y])$$

*whenever  $(a, x)$  and  $(b, y)$  are both either objects or morphisms in  $L'$ . Finally, let*

$$Y: (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

*be defined as follows:*

$$Y = (p \otimes p \otimes p) \circ J \circ j$$

*where  $p: L' \rightarrow L$  is the projection functor given by the fact that  $L' = K \oplus L$  and*

$$j: L \rightarrow L' \otimes L' \otimes L'$$

*is the linear functor defined by*

$$j(x) = (1, 0) \otimes (1, 0) \otimes (0, x),$$

*where  $x$  is either an object or morphism of  $L$ . Then  $Y$  is a solution of the Zamolodchikov tetrahedron equation if and only if  $J$  satisfies the Jacobiator identity.*

**Proof.** Equivalently, we must show that  $Y$  satisfies the Zamolodchikov tetrahedron equation if and only if  $J$  satisfies the Jacobiator identity. Applying the left-hand side of the Zamolodchikov tetrahedron equation to an object  $(a, w) \otimes (b, x) \otimes (c, y) \otimes (d, z)$  of  $L' \otimes L' \otimes L' \otimes L'$  yields an expression consisting of various uninteresting terms together with one involving

$$J_{[w,x],y,z}([J_{w,x,z}, y] + 1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}),$$

while applying the right-hand side produces an expression with the same uninteresting terms, but also one involving

$$[J_{w,x,y}, z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1)$$

in precisely the same way. Thus, the two sides are equal if and only if the Jacobiator identity holds. The detailed calculation resembles that of the proof of Proposition 64 and is quite lengthy.  $\square$

**Corollary 112.** *If  $L$  is a semistrict Lie 2-algebra, then  $Y$  defined as in Theorem 111 is a solution of the Zamolodchikov tetrahedron equation.*

This result could be just the beginning of a beautiful relationship between categorified Lie theory and higher-dimensional braids — a relationship that extends to higher and higher dimensions.

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